Geometric Complexity Theory and Orbit Closures of Homogeneous Forms

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Niemals aufgeben, niemals kapitulieren.
— Peter Quincy Taggart

Deutsche Einleitung

Das **P**-**NP**-Problem gehört zu den fundamentalsten und faszinierensten Problemen der heutigen Mathematik. Es hat umfangreiche Bedeutung für praktische Anwendungen und ist gleichermaßen eine grundsätzliche Frage über die Natur der Mathematik an sich. Wäre unerwartet $\mathbf{P} = \mathbf{NP}$, so könnte etwa ein Computer effizient bestimmen, ob eine mathematische Aussage wahr oder falsch ist. Seit die Frage 1971 von Cook [Coo71] gestellt wurde, scheinen wir einer Antwort jedoch nicht nennenswert näher gekommen zu sein. Der größte Fortschritt ist das ernüchternde Resultat von Razborov und Rudich [RR97], dass es keine „natürlichen" Beweise für $\mathbf{P} \neq \mathbf{NP}$ geben kann, siehe deren Arbeit für eine Definition und Details.

Peter Bürgisser hat gezeigt [Bü00], dass unter der verallgemeinerten Riemann-Hypothese die nicht-uniforme Version von $\mathbf{P} \neq \mathbf{NP}$ eine Vermutung von Valiant impliziert, welche weithin als ein algebraisches Analogon betrachtet wird. Diese Vermutung ist ein ebenso offenes Problem wie das ursprüngliche, doch es gibt die Hoffnung, dass die zusätzliche algebraische Struktur mehr Ansatzpunkte liefert. Wir geben einen kurzen Überblick über die zugrundeliegende Theorie in Kapitel 1: Die besagte Vermutung von Valiant (Vermutung 1.4.5) ist die zentrale Motivation für die hier vorgestellten Forschungsergebnisse.

In Kapitel 2 verschärfen wir ein Resultat von Valiant indem wir zeigen, dass sich jedes ganzzahlige Polynom stets als Determinante einer Matrix schreiben lässt, deren Einträge nur Variablen, Nullen und Einsen sind. The Größe der kleinsten solchen Matrix ist ein sinnvolles Komplexitätsmaß, welches sich rein kombinatorisch untersuchen lässt. Als Anwendung beweisen wir untere Schranken in kleinen Fällen durch Computerberechnung.

Von hier an widmen wir uns einem Ansatz zum Beweis von Valiant's Vermutung, welcher 2001 von Mulmuley und Sohoni vorgestellt wurde und den Titel „Geometric Complexity Theory" trägt, oder auch kurz GCT. Valiants Ergebnisse werden hier in Aussagen der Algebraischen Geometrie und Darstellungstheorie übersetzt, um dadurch die Probleme vorheriger Ansätze zu vermeiden [For09]. Eine Zusammenfassung dieses interessanten Übersetzungsprozesses ist in Kapitel 3 zu finden.

Um Komplexitätsklassen voneinander zu trennen, muss bewiesen werden, dass kein effizienter Algorithmus für ein mutmaßlich schweres Problem existiert. GCT liefert ein Kriterium, wonach die *Existenz* gewisser ganzzahliger Vektoren schon diese Trennung impliziert. Solche Vektoren nennen wir auch *Obstruktionen*. Die Hoffnung von Mulmuley und Sohoni war, dass man sich für den Beweis von Valiants Vermutung auf eine bestimmte Art von Obstruktionen beschränken kann, doch wir konnten bereits 2015 bemerken, dass es starke Anzeichen dagegen gibt. Wir stellen diese Ergebnisse in Kapitel 4 vor. Erst kürzlich haben Bürgisser, Ikenmeyer und Panova schlussendlich gezeigt, dass diese sogenannten *occurrence obstructions* nicht ausreichen, um Valiants Vermutung zu beweisen.

Im zweiten Teil der Arbeit beschäftigen wir uns näher und in größerer Allgemeinheit mit einem der zentralen Konzepte von GCT: Wir betrachten die Wirkung der GL_n auf dem Raum der homogenen, n-variaten Polynome vom Grad d durch Verkettung von rechts und beobachten, dass alle Elemente einer Bahn im Wesentlichen die gleiche Berechnungskomplexität haben. Man ordnet einem Polynom P das kleinste d zu, so dass P im Bahnabschluss des $d \times d$ – Determinantenpolynoms liegt. Diese Kennzahl ist dann äquivalent zu Valiants ursprünglichem Komplexitätsmaß für Polynome, sofern auch Approximationen zugelassen werden. Die Geometrie des Bahnabschlusses der Determinante ist wenig verstanden, sogar für kleine Werte von d. Wir können allerdings die im Rand auftretenden Komponenten für $d = 3$ vollständig klassifizieren.

Die benötigten algebraischen und geometrischen Werkzeuge werden im einführenden Kapitel 5 vorgestellt. Bahnabschlüsse von homogenen Formen sind stets algebraische Varietäten mit der oben genannten GL_n-Wirkung und werden klassisch in der geometrischen Invariantentheorie studiert. Neben der Determinante beschäftigen wir uns in den darauffolgenden Kapiteln auch mit den Bahnabschlüssen anderer homogener Formen.

In Kapitel 6 behandeln wir das allgemeine Monom $x_1 \cdots x_d$, dessen Relevanz für GCT daher stammt, dass es auch als Einschränkung des Determinantenpolynoms auf Diagonalmatrizen verstanden werden kann. Das Studium seines Bahnabschlusses ist beispielsweise das zentrale Hilfsmittel in Kapitel 4. Wir bemerken in diesem Kapitel, dass das Monom die seltene Eigenschaft hat, dass jedes Polynom in seinem Bahnabschluss lediglich das Ergebnis einer Variablensubstitution ist. Eine Klassifikation aller Polynome mit dieser Eigenschaft bleibt offen, obwohl wir einige Fragen im Hinblick darauf beantworten können.

Wir stellen für die beiden abschließenden Kapitel noch weitere Techniken in Kapitel 7 vor. Maßgeblich ist eine obere Schranke für die Anzahl irreduzibler Komponenten des Randes eines Bahnabschlusses. Die erste Anwendung dieser Aussage liefert eine Beschreibung des Randes für $\det_3$ in Kapitel 8. Wir bestimmen hier auch den Stabilisator der Determinante einer allgemeinen, spurlosen Matrix und können so schlussfolgern, dass der Bahnabschluss dieses Polynoms stets eine Komponente im Rand des Bahnabschlusses der Determinante ist.

Das letzte Kapitel enthält bisher unveröffentlichte Ergebnisse über das allgemeine Binom $x_1 \cdots x_d + y_1 \cdots y_d$. Wie auch das Monom ist das Binom im Bahnabschluss von $\det_d$ enthalten. Ein solides Verständnis dieser Familie homogener Formen sollte daher Voraussetzung dafür sein, den Bahnabschluss des Determinantenpolynoms im Allgemeinen zu studieren. Die hier aufkommenden geometrischen Fragen sind bereits deutlich komplexer als im Fall des Monoms: Wir können zwei Komponenten des Randes im Detail beschreiben, müssen jedoch eine subtile Frage offen lassen. Sofern diese sich positiv beantworten lässt, erhalten wir jedoch so bereits eine vollständige Beschreibung des Randes.

Contents

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Introduction

The **P** vs. **NP** problem is among the deepest and most intriguing questions of mathematics today. While having a multitude of implications for practical applications, it also carries fundamental questions in its wake, about the nature of mathematics itself. If for example $\mathbf{P} = \mathbf{NP}$ would hold unexpectedly, then we could program a machine to efficiently determine the truth of mathematical statements. Introduced in 1971 by Cook [Coo71], the problem is already 46 years old and we seem as far from the solution as ever. The most notable progress so far is the sobering result by Razborov and Rudich [RR97] that no "natural" proof for $\mathbf{P} \neq \mathbf{NP}$ exists, see their work for a definition and details.

Peter Bürgisser has shown [Bü00] that the nonuniform version of $\mathbf{P} \neq \mathbf{NP}$ implies, under the generalized Riemann hypothesis, a conjecture by Valiant which is widely considered an algebraic analogue. It remains as unresolved as the former, even though the additional algebraic structure involved is believed to provide better points of vantage. We give a brief review of the underlying theory in Chapter 1 – It is the said conjecture by Valiant (Conjecture 1.4.5) that serves as motivation for the research presented herein.

In Chapter 2, we slightly strengthen a result by Valiant: The core observation is that integer polynomials can always be written as the determinant of a matrix whose entries are variables, zeros and ones. The size of the smallest such matrix then gives a reasonable complexity measure which at the same time is accessible to combinatorics. As an application, we can provide lower bounds in small cases by computational methods.

From there on, we are concerned with a recent approach to Valiant's Conjecture known as Geometric Complexity Theory, or GCT for short. Introduced in 2001 by Mulmuley and Sohoni, it avoids the difficulties of many previous attempts [For09] by translating Valiant's results to statements in algebraic geometry and representation theory. An outline of this enticing transition is given in Chapter 3.

Where complexity theory is clasically concerned with proving the *nonexistence* of good algorithms for supposedly hard problems, GCT provides a criterion whereby the *existence* of certain integer vectors implies a separation of complexity classes. We refer to the vectors in question simply as *obstructions*. It was the hope of Mulmuley and Sohoni that the search could be further restricted to a particular kind of obstructions, but we already observed in 2015 that this appears unlikely – we present these results in Chapter 4. Quite recently in 2016 it was shown by Bürgisser, Ikenmeyer, and Panova that indeed these so-called occurrence obstructions do not suffice to prove Valiant's Conjecture, representing a bitter setback for the programme.

For the second part of the thesis, we treat a central topic of GCT in more detail and generality: One can measure the approximate complexity of polynomials by studying the closure of their orbit under the action of a general linear group by precomposition. The smallest d for which a polynomial appears in the orbit closure of the $d \times d$ determinant polynomial is equivalent to Valiant's original measure for its complexity, if approximations are permitted. The geometry of the determinant orbit closure is little understood even for small values of d, but we can give a classification of the components that appear in the boundary for $d = 3$.

The required toolbox of geometric and algebraic techniques is introduced in the preliminary Chapter 5. Quite generally, the orbit closure of a homogeneous form is an algebraic variety with a GL_n-action which is both intuitive and yet incredibly intriguing, and is an object of study to the beautiful fields of classical geometric invariant theory and birational geometry. In the subsequent chapters we study this problem for other homogeneous forms than the determinant.

Chapter 6 deals with a polynomial that has appeared in the context of GCT before, namely the universal monomial $x_1 \cdots x_d$. Its orbit closure is contained in the orbit closure of $\det_d$ and was instrumental in proving the results of Chapter 4, for example. It also has the remarkable and rare property that every polynomial in its orbit closure is the result of a variable substitution. A classification of all polynomials with this property remains open, but we both answer and pose several questions to advance it.

In Chapter 7, we introduce additional techniques used in the two subsequent chapters: We bound the number of irreducible components of the orbit closure boundary by the number of smooth blowups required to resolve the indeterminacy of a related rational map.

The first application of this technique yields a classification of the boundary of $\det_3$ in Chapter 8. Here, we also describe the stabilizer group of the determinant of a generic traceless matrix and conclude that the orbit closure of this polynomial is always a codimension one component of the boundary of the orbit closure of the determinant.

The final chapter contains unpublished work on the universal sum of two monomials, the binomial $x_1 \cdots x_d + y_1 \cdots y_d$. Like the monomial, the binomial is contained in the orbit closure of $\det_d$. A firm understanding of this polynomial should therefore precede the study of general determinantal expressions. The binomial already gives rise to a significantly more involved geometry than the monomial. We can describe two components of the boundary in detail, but must leave a subtle question open. If said question could be answered affirmatively however, these two components constitute the entire boundary of the orbit closure of the binomial.

Part I

Geometric Complexity Theory

Chapter 1

Algebraic Complexity Theory

The following is only the briefest of summaries: Complexity theory analyzes the complexity of solving *problems*. Problems have *instances* of different *sizes* which can be solved by *algorithms*. Denoting by $t(m)$ the minimum number of *steps* performed by any algorithm that solves all instances of size m, the *complexity* of a problem is the function $t\colon \mathbb{N} \to \mathbb{N}$. The definitions of *"problem"*, *"algorithm"* and related notions is encompassed by a *model of computation*. The classical model of computation is the Turing machine, which mimics our present-day computers. Liberally quoting the famous Church-Turing thesis, any problem that we consider computationally solvable is solvable by a Turing machine. This generality comes at a price, however: The Turing model provides very little mathematical structure to be exploited.

In this chapter, we explore the algebraic model, where a problem is given as a family of polynomial functions and the goal is quite simply to evaluate them.

1.1 Arithmetic Circuits

The algorithms in algebraic complexity theory are arithmetic circuits. An arithmetic circuit is a schematic representation of a way to compute a polynomial: Figure 1.1.1 shows a circuit computing $x^2 + xy \in \mathbb{Z}[x, y]$. In general, an arithmetic circuit is a directed, acyclic graph where at every vertex, a polynomial is being computed. Vertices with no ingoing edges contain constants or variables and vertices with exactly two ingoing edges are either labeled with the symbol "+" (Plus) or "×" (Times), computing the sum or the product of their input, respectively. This concept can be generalized to the notion of a *circuit*, which performs computation in an algebraic structure by associating to any vertex with k ingoing edges some k-ary operation.

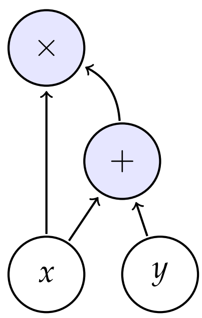


Figure 1.1.1: $x^2 + xy$

1.1.1 Definition. Let R be a commutative ring and $R[\mathbf{x}]$ the polynomial ring over R in a countably infinite set of variables $\mathbf{x}$. An **arithmetic circuit** over R is a directed acyclic graph C with vertex labels, subject to the following conditions:

(1) The vertices with no incoming edges are labelled with elements of $R \cup \mathbf{x}$. Any such vertex is called an **input gate**.

(2) Since C is acyclic, every vertex of C has a well-defined **depth**, which is the length of a longest path from an input gate to it. Any vertex of positive depth has exactly two incoming edges and is labelled with an element of $\{+, \times\}$. Any such vertex is also called a **computation gate**.

We define $P(v) \in R[\mathbf{x}]$ for every vertex v recursively by depth, as follows: If v is an input gate, $P(v)$ is defined to be the label of v. Otherwise, let $*_v \in \{+, \times\}$ be the label of v and denote by u and w the source vertices of the two incoming edges of v. Then, we can define $P(v) := P(u) *_v P(w)$.

The circuit C **computes** an element $P \in R[\mathbf{x}]$ if there is a vertex v with $P = P(v)$. The **size** of C is the number of computation gates, denoted by $|C|$. ●

1.1.2 Example. In Figure 1.1.2, we give an example of an arithmetic circuit over $\mathbb{C}$ which computes the equation of an affine elliptic curve in the variables $\{x, y\}$. ●

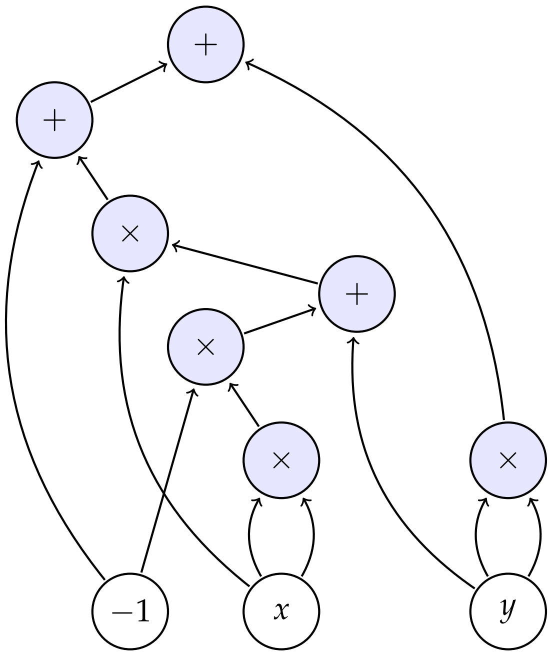
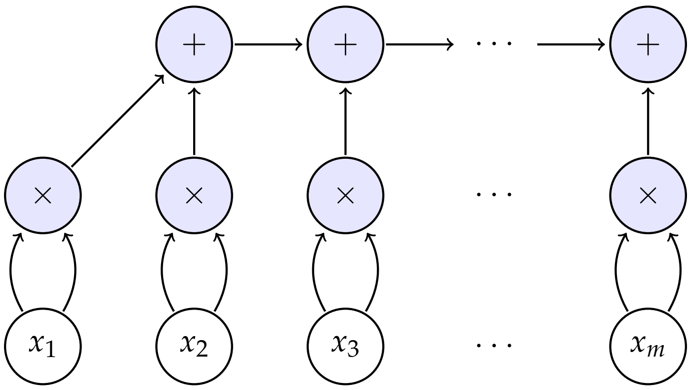


Figure 1.1.2: An arithmetic circuit computing $y^2 + xy - x^3 - 1$.

1.1.3 Definition. Let R be a commutative ring. The **(circuit) complexity** of a polynomial P with coefficients in R is the minimum size of an arithmetic circuit over R which computes P. We denote this number by $\mathrm{cc}^R(P)$. ●

It should be emphasized that complexity theory does not study the complexity of single polynomials. Instead, the object of study are *families* of polynomials. This is the main reason why it is convenient to work with infinitely many variables.

1.1.4 Example. Consider the polynomial $P_m := x_1^2 + \cdots + x_m^2 \in \mathbb{R}[\mathbf{x}]$ for $m \in \mathbb{N}$. We study the map $\mathbb{N} \to \mathbb{N}$ given by $m \mapsto \mathrm{cc}^{\mathbb{R}}(P_m)$. We claim that $\frac{1}{2}m \le \mathrm{cc}^{\mathbb{R}}(P_m) \le 2m$, which is interpreted by saying that the complexity of computing a Euclidean norm is linear in the input size. It is easy to see that the circuit



computes P_m with $2m-1$ computation gates. Since P_m is supported on m variables, each of which needs to be connected to some computation gate in a circuit computing P_m, we can see that $\mathrm{cc}^{\mathbb{R}}(P_m) \ge \frac{1}{2}m$ because at most two variables can be connected to the same gate. With a more refined argument, one can actually show $\mathrm{cc}^{\mathbb{R}}(P_m) \ge m$ in an even stronger model of computation [BCS97, Example 6.1]. ●

1.1.5 Notation. We fix a commutative ring R and a countably infinite set $\mathbf{x}$ of variables. A polynomial will always be an element of $R[\mathbf{x}]$ unless stated explicitly otherwise. We will also write $P \in R[x_1, \ldots, x_n]$ for a polynomial in n variables and implicitly identify the x_i with certain elements of $\mathbf{x}$.

> The elements of $\mathbf{x}$ are not sequentially numbered or named in *any way*, a priori. We do not assume any ordering on $\mathbf{x}$. We may often write $x_1, \ldots, x_r \in \mathbf{x}$, but we may also write $a, b, c \in \mathbf{x}$ or $y \in \mathbf{x}$.

This way, we always operate in the ring $R[\mathbf{x}]$, regardless of what (finite) number of variables we require. Finally, we write $\mathrm{cc}(P)$ instead of $\mathrm{cc}^R(P)$ unless we want to put emphasis on the ring R. ●

Remark. In Chapter 2, R will be equal to $\mathbb{Z}$ and starting with Chapter 3, R will be the field of complex numbers. ○

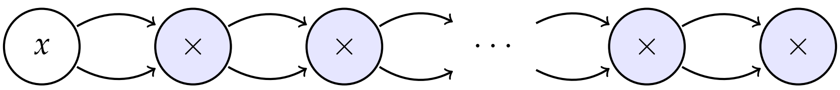
1.2 The Classes VP and P

Only for very few and rather simple families $(P_m)_{m\in\mathbb{N}}$ of polynomials, we can determine the function $m \mapsto \mathrm{cc}(P_m)$ "explicitly". As is often the case in complexity theory, we restrict instead to the classification of such functions by their asymptotic rate of growth. Even this task turns out to be quite challenging.

1.2.1 Definition. A function $t\colon \mathbb{N} \to \mathbb{N}$ is called **polynomially bounded** if there is a polynomial $p \in \mathbb{Z}[x]$ and $m_0 \in \mathbb{N}$ such that $\forall m \geq m_0\colon t(m) \leq p(m)$. We define poly as the set of all polynomially bounded functions $\mathbb{N} \to \mathbb{N}$.

Remark. We will often write $t(m) \in \mathrm{poly}(m)$ instead of $t \in \mathrm{poly}$, for example we would write $m^3 \in \mathrm{poly}(m)$ instead of giving the function $m \mapsto m^3$ a name. If a numeric quantity $t(m) \in \mathbb{N}$ is associated to each element of a family $P = (P_m)_{m\in\mathbb{N}}$, we say that P has (or admits) **polynomially many** of said quantity to express that $t \in \mathrm{poly}$.

It turns out that it is not desirable to study arbitrary families of polynomials. For example, the value x^{2^m} for $x \in \mathbb{Z}$ cannot be computed by a Turing machine in polynomial time simply because the output is too large, but the circuit



computes it with only m computation gates.

1.2.2 Definition. We say that P is a **p-family** if $P = (P_m)_{m\in\mathbb{N}}$ is a family of polynomials with $\deg(P_m) \in \mathrm{poly}(m)$. The complexity class **VP** is defined to be the set of all p-families P with $\mathrm{cc}(P_m) \in \mathrm{poly}(m)$.

Remark. Although it is not visible in the notation, this definition depends on the coefficient ring R.

VP is the circuit analogue of the complexity class **P**. The "V" in its name stands for the name of its inventor **V**aliant, who introduced these notions in [Val79a].

1.2.1 The Class P

Classical complexity theory deals with families of boolean functions $\mathbb{F}_2^m \to \mathbb{F}_2$. A family $B_m\colon \mathbb{F}_2^m \to \mathbb{F}_2$ is in the complexity class **P** iff there exists a Turing machine which computes B_m and requires polynomially many steps to do so. We do not give the formal definition of a Turing machine, but it may be thought of as a computer algorithm, where a step is a physically atomic operation for the machine.

The class **P** has the *nonuniform* analogue **P**/ poly which has a description in terms of circuits: Nonuniformity means that we are not asking for a single algorithm to work on all input sizes, but we allow different algorithms for different input sizes. In the language of circuits, a family $B_m \colon \mathbb{F}_2^m \to \mathbb{F}_2$ of boolean functions is in **P**/ poly if and only if $\mathrm{cc}^{\mathbb{F}_2}(B_m) \in \mathrm{poly}(m)$. Note that by polynomial interpolation, any function $\mathbb{F}_2^m \to \mathbb{F}_2$ is given by some polynomial in $\mathbb{F}_2[x_1, \ldots, x_m] \subseteq \mathbb{F}_2[\mathbf{x}]$.

1.3 Reduction and Completeness

An important concept in boolean as well as algebraic complexity theory is reduction. Informally put, a problem Q can be reduced to a problem P if one can produce an algorithm for Q from an algorithm for P without measurably increasing the runtime. This is only a vague description, of course. We will give the precise definition for the algebraic model after introducing some notation.

1.3.1 Definition. Denote by $\mathbb{N}^{(\mathbf{x})}$ the $\mathbf{x}$-indexed sequences $\alpha = (\alpha_x)_{x \in \mathbf{x}}$ of natural numbers with $\alpha_x \neq 0$ for only finitely many $x \in \mathbf{x}$. By definition, a polynomial $P \in R[\mathbf{x}]$ is a map $P \colon \mathbb{N}^{(\mathbf{x})} \to R$ which we denote $\alpha \mapsto P_\alpha$ and require $P_\alpha \neq 0$ for only finitely many $\alpha \in \mathbb{N}^{(\mathbf{x})}$. One then writes

$$P = \sum_{\alpha \in \mathbb{N}^{(\mathbf{x})}} P_\alpha \cdot \prod_{x \in \mathbf{x}} x^{\alpha_x}.$$

We define the **support** of P to be the set of variables that occur in this expression, i.e.

$$\mathrm{supp}(P) := \left\{ x \in \mathbf{x} \,\middle|\, \exists \alpha \in \mathbb{N}^{(\mathbf{x})} \colon \alpha_x \neq 0 \wedge P_\alpha \neq 0 \right\}.$$

Note that $\mathrm{supp}(P)$ is always a finite set. For $P \in R[\mathbf{x}]$ with $\mathrm{supp}(P) = \{x_1, \ldots, x_n\}$, the expression for P becomes the familiar $P = \sum_{\alpha \in \mathbb{N}^n} P_\alpha \cdot \prod_{i=1}^n x_i^{\alpha_i}$. ●

1.3.2 Example. Let $x, y, z \in \mathbf{x}$. For $Q = x^2 y + z^3$, we have $\mathrm{supp}(Q) = \{x, y, z\}$. The polynomial $P = x^2 y + y^3$ can also be viewed as an element of $\mathbb{C}[x, y, z]$ but we have $\mathrm{supp}(P) = \{x, y\}$ because z does not occur in P. ●

1.3.3 Remark. If $P \in \mathbf{VP}$, then $|\mathrm{supp}(P_m)| \in \mathrm{poly}(m)$. In other words, any family in **VP** requires only polynomially many variables. Indeed, in a circuit of minimum size computing P_m, every input gate containing a variable is connected to at least one computation gate and each computation gate can be connected to at most two input gates. Since there are only polynomially many computation gates, there can only be polynomially many variables in a (nonredundant) expression for P_m. ●

We next define a partial order "$\leq$" on $R[\mathbf{x}]$ where $P \leq Q$ holds if P can be obtained from Q by variable substitution. For example, $P \leq Q$ holds in Example 1.3.2 because P arises from Q by substituting z for y. This is formalized in the following Definition 1.3.4:

1.3.4 Definition. For $S \subseteq \mathbf{x}$ and a map $\sigma\colon S \to R[\mathbf{x}]$, we extend σ to a map $\mathbf{x} \to R[\mathbf{x}]$ by the identity. We then denote by P^σ the image of P under the unique R-algebra homomorphism $R[\mathbf{x}] \to R[\mathbf{x}]$ which maps $x \mapsto \sigma(x)$ for all $x \in \mathbf{x}$.

A polynomial P is called a **projection** of another polynomial Q if there is some $S \subseteq \operatorname{supp}(Q)$ and a map $\sigma\colon S \to \mathbf{x} \cup R$ such that $P = Q^\sigma$. We denote this by $P \leq_R Q$. The definition extends to families as follows: A p-family P is a **p-projection** of another p-family Q, in symbols $P \leq_R Q$, if there is some $t \in \mathrm{poly}$ such that $P_m \leq Q_{t(m)}$ for all $m \in \mathbb{N}$. We will write $P \leq Q$ in both cases if there is no ambiguity concerning R. ●

1.3.5 Example. Let $x, y, z \in \mathbf{x}$ and $Q := y^2z + xyz - x^3 - z^3$. Let $\sigma\colon \mathbf{x} \to R[\mathbf{x}]$ be the identity everywhere except for $\sigma(z) = 1$. Then, one obtains $Q^\sigma = y^2 + xy - x^3 - 1$. ●

1.3.6 Example. Let $a_i \in \mathbf{x}$ and $x_i \in \mathbf{x}$ be distinct variables for all $i \in \mathbb{N}$. For $d \in \mathbb{N}$, we consider the family $Q = (Q_d)_{d \in \mathbb{N}}$ given by

$$Q_d := a_0 + a_1x_1 + \ldots + a_dx_d$$

with $\operatorname{supp}(Q_d) = \{a_0, \ldots, a_d, x_1, \ldots, x_d\}$. Any p-family $P \in \mathbf{VP}$ of affine linear polynomials satisfies $P \leq Q$. Indeed, this follows because $t(m) := |\operatorname{supp}(P_m)|$ is polynomially bounded by Remark 1.3.3. ●

1.4 The Classes VNP and NP

Valiant introduced the class **VNP** as an analogue of the class **NP**, but we will give a definition of the class **VNP** first because our focus is on the algebraic model. We will then discuss the relation to **NP**.

1.4.1 Definition. Let P be a p-family. Then $P \in \mathbf{VNP}$ if and only if there exists a family $Q \in \mathbf{VP}$ and a sequence $(C_m)_{m \in \mathbb{N}}$ with $C_m \subseteq \operatorname{supp}(Q_m)$ such that P is a projection of the family $\tilde{Q} = (\tilde{Q}_m)_{m \in \mathbb{N}}$, which is defined as

$$\tilde{Q}_m := \sum_{\sigma\colon C_m \to \{0,1\}} Q_m^\sigma \cdot \prod_{\substack{x \in C_m \\ \sigma(x)=1}} x.$$

●

Remark. We use here the original definition from [Val79a, p. 252] because we can later make the analogy to classical complexity classes more clear. In subsequent literature, the definition appears in seemingly weaker, but equivalent form [Bü00, p. 5]. ○

Remark. It follows from the definition that **VP** $\subseteq$ **VNP**: For $P \in$ **VP**, choose $Q = P$ and $C_m := \emptyset$ for all $m \in \mathbb{N}$. By Definition 1.3.1, the empty map $\sigma\colon \emptyset \to \{0,1\}$ satisfies $P_m^\sigma = P_m$ for all $m \in \mathbb{N}$. ∘

Remark. Note that for $P \in$ **VNP**, we again have $|\mathrm{supp}(P_m)| \in \mathrm{poly}(m)$. Indeed, in Definition 1.4.1 we know that $|\mathrm{supp}(Q_m)| \in \mathrm{poly}(m)$ by Remark 1.3.3. Furthermore, $\mathrm{supp}(\tilde{Q}_m) \subseteq \mathrm{supp}(Q_m)$ and and since $P_m \leq \tilde{Q}_m$, we have

$$|\mathrm{supp}(P_m)| \leq |\mathrm{supp}(\tilde{Q}_m)| \leq |\mathrm{supp}(Q_m)| \in \mathrm{poly}(m).$$ ∘

1.4.1 The Class NP

We will now explain and motivate this definition by comparing Valiant's class with the classical one. A family $B_m\colon \mathbb{F}_2^m \to \mathbb{F}_2$ is in **NP** if and only if there are functions $t_m \in \mathrm{poly}(m)$ and $C_m\colon \mathbb{F}_2^m \times \mathbb{F}_2^{t_m} \to \mathbb{F}_2$ such that

- the family $(C_m)_{m\in\mathbb{N}}$ is in **P** and
- $\forall m \in \mathbb{N}\colon \forall \sigma \in \mathbb{F}_2^m\colon\ (B_m(\sigma) = 1) \Leftrightarrow (\exists c \in \mathbb{F}_2^{t_m}\colon C_m(\sigma, c) = 1)$.

In other words, it might not be easy to decide whether $B_m(\sigma) = 1$, but it is easy to confirm that $B_m(\sigma) = 1$ if given a valid certificate $c \in \mathbb{F}_2^{t_m}$.

Again, there is a relevant related complexity class known as **#P**. It contains families of maps $\mathbb{F}_2^m \to \mathbb{N}$. Such a family $P_m\colon \mathbb{F}_2^m \to \mathbb{N}$ is in **#P** if and only if there exists a family $(B_m)_{m\in\mathbb{N}} \in$ **NP** which has a certificate function $(Q_m)_{m\in\mathbb{N}} \in$ **P** such that P_m counts the number of certificates, i.e.,

$$P_m(\sigma) = \left|\left\{ c \in \mathbb{F}_2^{t_m} \,\middle|\, Q_m(\sigma, c) = 1 \right\}\right|.$$

Note that $B_m(\sigma) = 1$ if and only if $P_m(\sigma) > 0$.

If we interpret $Q_m \in \mathbb{F}_2[\mathbf{x}, \mathbf{y}]$ where $\mathbf{x} = \{x_1, \ldots, x_m\}$ and $\mathbf{y} = \{y_1, \ldots, y_{t_m}\}$, then

$$\tilde{Q}_m := \sum_{\sigma\colon \mathbf{y} \to \{0,1\}} Q_m^\sigma \cdot \prod_{\substack{k=1 \\ \sigma(y_k)=1}}^{t_m} y_k$$

is an element of **VNP** over $\mathbb{F}_2$ by Definition 1.4.1 and for any $\sigma \in \mathbb{F}_2^m = \mathbb{F}_2^{\mathbf{x}}$, i.e., viewing σ as a map $\sigma\colon \mathbf{x} \to \{0,1\} = \mathbb{F}_2$, we have $\tilde{Q}_m^\sigma = 0$ if and only if $B_m(\sigma) = 0$. More precisely, $\tilde{Q}_m^\sigma$ has exactly $P_m(\sigma)$ monomials and the monomial

$$\prod\nolimits_{\substack{k=1 \\ c_k=1}}^{t_m} y_k$$

occurs in $\tilde{Q}_m^\sigma$ if and only if $c \in \mathbb{F}_2^{t_m}$ is a certificate with $Q_m(\sigma, c) = 1$.

1.4.2 Completeness of the Permanent

1.4.2 Definition. Let $\mathcal{C}$ be some class of p-families, like **VP** or **VNP**. A p-family P is called $\mathcal{C}$-**complete** if $P \in \mathcal{C}$ and $\forall Q \in \mathcal{C}\colon\ Q \leq P$. ●

Two completeness results by Valiant make the connection between #**P** and **VNP** even more tangible. They both concern the permanent polynomial family. The **permanent** of an $m \times m$ matrix $(x_{ij})_{1 \leq i,j \leq m}$ is defined as

$$\mathrm{per}_m := \sum_{\pi \in \mathfrak{S}_m} \prod_{i=1}^{m} x_{i,\pi(i)}, \tag{1}$$

where $\mathfrak{S}_m$ denotes the symmetric group on m symbols, i.e., the group of all set automorphisms of $[m] := \{1, \ldots, m\}$. We will consider $\mathrm{per} = (\mathrm{per}_m)_{m \in \mathbb{N}}$ as a family of polynomials in the variables x_{ij}.

1.4.3 Theorem ([Val79b]). The problem of computing the permanent of a binary matrix (one where every entry is either 0 or 1) is #**P**-complete.

1.4.4 Theorem ([Val79a]). The permanent family is **VNP**-complete over any field of characteristic different from 2.

By a formula of Ryser [Rys63], we know that per_m can be computed by a circuit of size $m \cdot 2^m$, but no significant improvement over this is known. This leads to the following conjecture:

1.4.5 Conjecture (Valiant's Hypothesis). The inclusion **VP** $\subseteq$ **VNP** is strict.

Given any **VP**-complete family P, one could prove **VP** $\neq$ **VNP** by showing that the permanent is not a p-projection of P.

1.5 Determinant Versus Permanent

The definition of the permanent (1) is quite similar to the familiar definition of the **determinant** family

$$\det_d := \sum_{\pi \in \mathfrak{S}_d} \mathrm{sgn}(\pi) \cdot \prod_{i=1}^{d} x_{i,\pi(i)}.$$

It is known [MV97] that $\det_d$ can be computed by a circuit of size $\mathcal{O}(d^6)$ – this is not immediately obvious because the classical procedure of Gaussian elimination performs divisions: Arithmetic circuits only allow for multiplication and addition.

Unfortunately, the determinant is not known to be **VP**-complete, nor is it thought to be. However, it is complete for a subclass of **VP** which we will study next.

1.5.1 Classes for the Determinant

The class of polynomials for which the determinant family is complete can be defined in several different ways. We will not go into detail and instead refer to [MP08; Tod92]. However, we give one definition:

An arithmetic circuit C is **weakly skew** if every vertex with label "$\times$" has one incoming edge which is a *bridge*, i.e., removing this edge increases the number of connected components of C.

1.5.1 Example. Recall the circuit computing $y^2 + xy - x^3 - 1$ from Example 1.1.2. It is not weakly skew, the condition is for example violated at the topmost multiplication gate computing $xy - x^3 - 1$. However, the circuit can be made weakly skew by duplicating some of the input gates: See Figure 1.5.1. At each multiplication gate, the input edge which is a bridge has been highlighted.

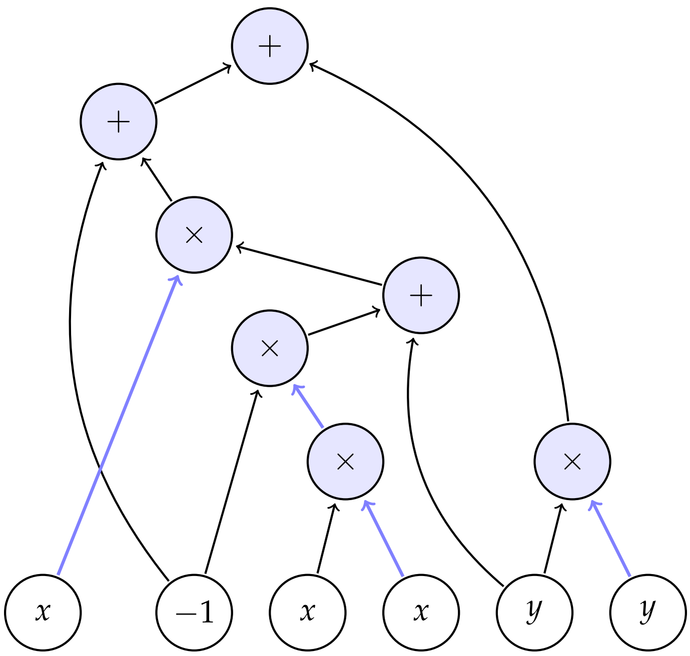


Figure 1.5.1: A weakly skew arithmetic circuit computing $y^2 + xy - x^3 - 1$.

Denote by $\mathrm{cc}_{\mathrm{ws}}(P)$ the minimum number d such that P can be computed by a weakly skew circuit of size d. $\mathbf{VP}_{\mathbf{ws}}$ consists of all p-families P such that $\mathrm{cc}_{\mathrm{ws}}(P_m) \in \mathrm{poly}(m)$.

1.5.2 Theorem ([Tod92],[MP08, Lemma 6]). The determinant family is $\mathbf{VP}_{\mathbf{ws}}$-complete and for any $P \in R[\mathbf{x}]$ with $d := \mathrm{cc}_{\mathrm{ws}}(P)$, we have $P \leq \det_{d+1}$. ■

Clearly, $\mathbf{VP}_{\mathbf{ws}} \subseteq \mathbf{VP}$ and therefore, if Valiant's Hypothesis (Conjecture 1.4.5) holds, the inclusion $\mathbf{VP}_{\mathbf{ws}} \subseteq \mathbf{VNP}$ must also be strict. Since the determinant is $\mathbf{VP}_{\mathbf{ws}}$-complete and the permanent is $\mathbf{VNP}$-complete by Theorem 1.4.4, $\mathbf{VP}_{\mathbf{ws}} \neq \mathbf{VNP}$ is equivalent to the following:

1.5.3 Conjecture (Permanent vs. Determinant). The permanent polynomial family is not a p-projection of the determinant polynomial family.

1.5.2 Determinantal Complexity

In order to approach Permanent versus Determinant (Conjecture 1.5.3), we need to study p-projections of the determinant. We will show that the determinant is universal, meaning that every polynomial is a projection of $\det_d$ for some $d \in \mathbb{N}$. This important insight gives rise to the definition of **determinantal complexity**: For a polynomial P we define

$$\mathrm{dc}(P) := \min\{d \in \mathbb{N} \mid P \leq \det_d\}.$$

This allows us to reformulate Conjecture 1.5.3 simply as $\mathrm{dc}(\mathrm{per}_m) \notin \mathrm{poly}(m)$. The benefit of this reformulation is the fact that there are no more circuits involved in it and methods from algebra directly apply.

1.5.4 Theorem ([Val79a, §2]). Let R be a commutative ring and $P \in R[x_1, \dots, x_n]$ a polynomial. Then, there is a natural number $d \in \mathbb{N}$ such that P is a projection of $\det_d$.

Remark. This highlights the importance of the determinant in a very fundamental way. Quoting Valiant himself, *"for the problem of finding a subexponential formula for a polynomial when one exists, linear algebra is essentially the only technique in the sense that it is always applicable."* ○

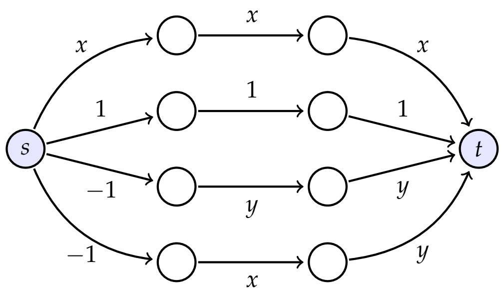
Proof. The proof is based on a graph construction which we will sketch here: Given a matrix $a = (a_{ij})_{1 \leq i,j \leq m}$ we can consider the labelled directed graph G_a on the set of vertices $\{1, \dots, m\}$ where the edge (i,j) has label a_{ij}. We treat edges with label 0 as nonexistent. For a directed graph with labels in $R[x_1, \dots, x_n]$, we can reverse this process and obtain a matrix from G which we will refer to as its *adjacency matrix*. A *cycle cover* of G_a is a partition of G_a into vertex disjoint cycles. The permutations on the set $\{1, \dots, m\}$ are in bijection with the cycle covers of G_a. To each cycle we associate a sign, which is -1 if the cycle has even length and 1 otherwise. To each cycle we then associate a weight, which is its sign times the product of its edge labels. The weight of a cycle cover is the product of the weights of its cycles. The determinant of a is then, by definition, the sum of the weights of all cycle covers of G_a.

Given a polynomial $P \in R[x_1, \dots, x_n]$, let $k := \deg(P)$ be its (total) degree. A product of $k+1$ constants and variables can produce any monomial that occurs in P, so we can write $P = \sum_{i=1}^{r} \prod_{j=0}^{k} P_{ij}$ for some $r \in \mathbb{N}$ and $P_{ij} \in R \cup \{x_1, \dots, x_n\}$. For example, consider the polynomial

$$P = x^3 + 1 - y^2 - xy = (x \cdot x \cdot x) + (1 \cdot 1 \cdot 1) + (-1 \cdot y \cdot y) + (-1 \cdot x \cdot y).$$

For this example, we used only 3 factors in each summand as opposed to the 4 factors that the construction would yield for a general cubic polynomial.

We construct a graph H as follows: For each of the r summands in the above representation of P, consider a path with $k+1$ edges where the edges of path i are labeled with the values p_{ij} for $0 \leq j \leq k$.



We identify the start vertices of all these paths and call it s, then we identify the end vertices of all these paths and call it t. We obtain an acyclic graph with r paths going from s to t.

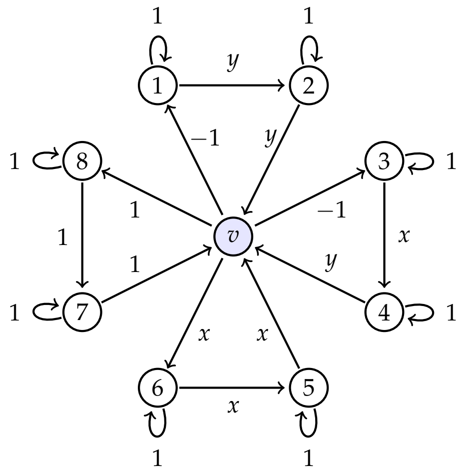


Figure 1.5.2: The Graph G

Let then G be the graph that arises from H by first adding loops with label 1 to all vertices except s and t and then identifying s and t into a single vertex v.

By construction, every cycle in G which is not a loop contains the vertex v and all these cycles have the same length. Therefore, the cycle covers of G are in bijection with the summands $\prod_{j=0}^{k} p_{ij}$ via their weight, up to a common sign. It follows that the adjacency matrix a of G satisfies $\det(a) = \pm P$. If $\det(a) = -P$, we can achieve $\det(a) = P$ by considering the block matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & -1 \end{smallmatrix}\right)$ instead of a. ■

Remark. Treating the vertex v in our example as the 9-th vertex and numbering all other vertices as in Figure 1.5.2, one can check that indeed

$$\det \begin{pmatrix} 1 & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 1 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & x & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & x & 0 & 1 & 0 \end{pmatrix} = x^3 + 1 - y^2 - xy.$$

However, this procedure does not yield the optimal result. For example,

$$\det \begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ x & y & y & 1 \end{pmatrix} = x^3 + 1 - y^2 - xy.$$

We used computational methods akin to the algorithm from the next chapter to compute this representation. ∘

Chapter 2

Binary Determinantal Complexity

This chapter contains the contents of the previously published work [HI16]. We only consider polynomials with integer coefficients here, i.e., we assume $R = \mathbb{Z}$.

Proving Valiant's Hypothesis (Conjecture 1.4.5) amounts to bounding the growth of $\mathrm{dc}(\mathrm{per}_m)$ superpolynomially from below. Lower bounds are a notoriously difficult problem in complexity theory. On the other hand, finding upper bounds admits the straightforward approach of constructing algorithms: In our case, an algorithm can be described as a sequence of matrices $A_m \in (\mathbf{x} \cup \mathbb{Z})^{t_m \times t_m}$ such that $\det(A_m) = \mathrm{per}_m$. Here, the numbers $t_m \geq \mathrm{dc}(\mathrm{per}_m)$ achieve equality if and only if the algorithm is optimal. To get a better idea of how $\mathrm{dc}(\mathrm{per}_m)$ grows, it is therefore reasonable to attempt the construction of good algorithms. The best one known so far is a graph construction by Grenet [Gre11], see Section 2.5, with the following consequence:

2.0.1 Theorem. For every natural number m there exists a matrix A of size $2^m - 1$ such that $\mathrm{per}_m = \det(A)$. Moreover, A can be chosen such that the entries in A are only variables, zeros, and ones, but no other constants. ■

Theorem 2.0.1 gives rise to the following definition: We call a matrix whose entries are only zeros, ones, or variables, a **binary variable matrix**. We will prove in Corollary 2.1.3 that every polynomial P with integer coefficients can be written as the determinant of a binary variable matrix and that the size is almost equal to $\mathrm{dc}(P)$, see Proposition 2.1.2 for a precise statement. We then denote by $\mathrm{bdc}(P)$ the smallest d such that P can be written as a determinant of an $d \times d$ binary variable matrix. It is called the **binary determinantal complexity** of P.

The complexity class of families $(P_m)_{m \in \mathbb{N}}$ with polynomially bounded binary determinantal complexity $\mathrm{bdc}(P_m)$ is exactly $\mathbf{VP}^0_{\mathbf{ws}}$, the constant free version of $\mathbf{VP}_{\mathbf{ws}}$, see Section 2.4 for definitions and proofs.

Theorem 2.0.1 shows that $\mathrm{bdc}(\mathrm{per}_m) \leq 2^m - 1$. This upper bound is clearly sharp for $m = 1$ and for $m = 2$, and we can also verify that it is sharp for $m = 3$:

2.0.2 Theorem. $\mathrm{bdc}(\mathrm{per}_3) = 7$.

We use a computer aided proof and enumeration of bipartite graphs in our study. The binary determinantal complexity of per_m is now known to be exactly $2^m - 1$ for $m \in \{1, 2, 3\}$. Unfortunately, determining $\mathrm{bdc}(\mathrm{per}_4)$ is currently out of reach with our methods.

The best known general lower bound is $\mathrm{bdc}(\mathrm{per}_m) \geq \mathrm{dc}(\mathrm{per}_m) \geq \frac{m^2}{2}$ due to [MR04] in a stronger model of computation, see also [LMR13] for the same bound in an even stronger model of computation. After the proof of Theorem 2.0.2 was published, Alper, Bogart, and Velasco proved in [ABV15] that $\mathrm{dc}(\mathrm{per}_3) = 7$, but unfortunately per_4 remains out of reach even with their methods.

2.1 The Cost of Computing Integers

The main purpose of this section is to prove that even though we only allow the constants 0 and 1, *all* polynomials with integer coefficients can be obtained as the determinant of a binary variable matrix, see Corollary 2.1.3. Moreover, the size of the matrices is not much larger than had we allowed integer constants, see Proposition 2.1.2. We use standard techniques from algebraic complexity theory, heavily based on [Val79a], but a certain attention to the signs has to be taken.

In what follows, a **digraph** is always a finite directed graph which may possibly have loops, but which has no parallel edges. We label the edges of a digraph by polynomials. We will almost exclusively be concerned with digraphs whose labels are only variables or the constant 1. Note that we consider only labeled digraphs.

A cycle cover of a digraph G is a set of cycles in G such that each vertex of G is contained in exactly one of these cycles. If a cycle in G has i edges with labels $e_1, \ldots, e_i$, then its **weight** is defined as $(-1)^{i-1} \cdot e_1 \cdots e_i$. The weight of a cycle cover is the product of the weights of its cycles. The **value** of G is the polynomial that arises as the sum over the weights of all cycle covers in G. We then define the *directed adjacency matrix* A of a digraph G as the matrix whose entry A_{ij} is the label of the edge (i, j) or 0 if that edge does not exist.

In what follows, we will often construct matrices as the directed adjacency matrices of digraphs. The reason is the well-known observation that the value of a digraph G equals the determinant of its directed adjacency matrix – see for example [Val79a].

As an intermediate step, we will often construct a *binary algebraic branching program*: This is an acyclic digraph $\Gamma = (\Gamma, s, t)$ where every edge is labeled by either 1 or a variable. The digraph Γ has two distinguished vertices, the *source* s and the *target* t, where s has no incoming and t has no outgoing edges. If an s-t-path in Γ has i edges with labels $e_1, \ldots, e_i$, then its **path weight** is defined as the value $(-1)^{i-1} \cdot e_1 \cdots e_i$. The **path value** of Γ is the polynomial that arises as the sum over the path weights of all s-t-paths in Γ. We remark that this notion of weight differs from the literature by a sign.

2.1.1 Proposition. For a nonzero constant $c \in \mathbb{Z}$, there is a binary algebraic branching program Γ with at most $\mathcal{O}(\log |c|)$ vertices whose path value is c.

Proof. We can assume without loss of generality that $c > 0$: Given a binary algebraic branching program Γ with path value $c > 0$ and at most $\mathcal{O}(\log c)$ vertices, we can add a single vertex t' and an edge from t to t' with label 1 to obtain a new program (Γ', s, t') with path value $-c$.

For a natural number c, an *addition chain* of length ℓ is a sequence of distinct natural numbers $1 = c_0, c_1, \ldots, c_\ell = c$ together with a sequence of tuples $(j_1, k_1), \ldots, (j_\ell, k_\ell)$ such that $c_i = c_{j_i} + c_{k_i}$ and $j_i, k_i < i$ for all $1 \leq i \leq \ell$. However, we will think of this data as a digraph $\tilde{\Gamma}$ on the vertices $\{v_0, \ldots, v_\ell\}$ with edges (v_{j_i}, v_i) and (v_{k_i}, v_i) for all $1 \leq i \leq \ell$. The labels of all edges are equal to 1. Note that we allow double edges in these digraphs temporarily. We set $s := v_0$ and $t := v_\ell$. Thus, we view an addition chain as an acyclic digraph where every vertex except for v_0 has indegree two. This already strongly resembles a binary algebraic branching program, but $\tilde{\Gamma}$ might have parallel edges. Observe that there are exactly c_i many paths from v_0 to v_i in the digraph $\tilde{\Gamma}$. In particular, there are exactly c paths from s to t in $\tilde{\Gamma}$.

Using the algorithm of repeated squaring [Knu98, Sec. 4.6.3, eq. (10)] one can construct an addition chain $\tilde{\Gamma}$ as above with at most $\mathcal{O}(\log c)$ vertices and such that there are exactly c paths from s to t in $\tilde{\Gamma}$. For every edge (v, w) in $\tilde{\Gamma}$ we add a new vertex u and replace the edge (v, w) by two new edges (v, u) and (u, w). We call the resulting digraph $\Gamma = (\Gamma, s, t)$. Observe that the binary algebraic branching program Γ has no parallel edges any more and all s-t-paths in Γ have even length. Also, the digraph Γ still has $\mathcal{O}(\log c)$ many vertices. Labelling all edges in Γ with 1, the path value of Γ is equal to c. ∎

2.1.2 Proposition. Let C be a $d \times d$ matrix whose entries are variables and *arbitrary* integer entries. Let $c_{\max}$ be the integer entry of C with the largest absolute value. Then there is a binary variable matrix A of size $\mathcal{O}(d^2 \cdot \log |c_{\max}|)$ with $\det(A) = \det(C)$.

Proof. We will interpret C as the directed adjacency matrix of a digraph. Any edge that has an integer label which is neither 1 nor 0 will be replaced by a subgraph of size $\mathcal{O}(\log |c_{\max}|)$ arising from the construction of the previous Proposition 2.1.1. The directed adjacency matrix of the resulting graph will be the desired matrix A. Formally, we proceed by induction.

Denote by k the number of integer entries in the matrix C that are neither equal to 0 nor 1. By induction on k, we will prove the slightly stronger statement that there

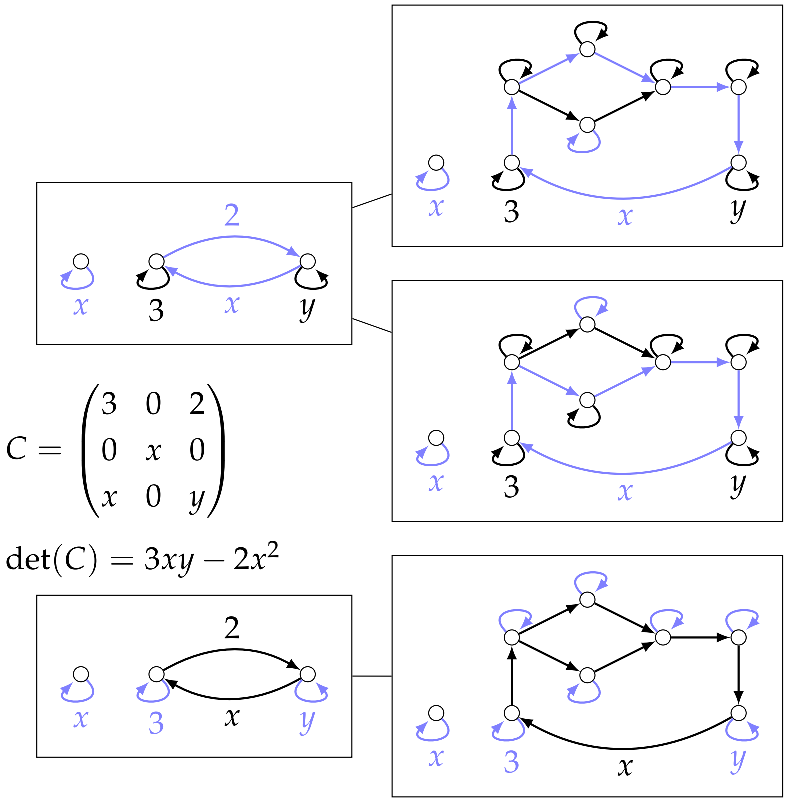


Figure 2.1.1: Given a matrix C we construct a digraph H with directed adjacency matrix C (left hand side) and the digraph G (right hand side) by replacing the edge with label 2 in H by a binary algebraic branching program. We omit the labels for edges that have label 1. The right hand side depicts the cycle covers K of G and the left hand side shows the corresponding cycle covers K^H of H.

is a binary variable matrix A of size $d + k \cdot \mathcal{O}(\log |c_{\max}|)$ with $\det(A) = \det(C)$. Since $k \leq d^2$, this implies the statement. Note that the case $k = 0$ is trivial, so we assume $k \geq 1$ and perform the induction step.

Let H be the digraph whose directed adjacency matrix is C. Recall that this means the following: H is a digraph on the vertices $1, \ldots, d$ and there is an edge (i, j) with label C_{ij} if $C_{ij} \neq 0$ and otherwise no such edge exists. Let $e = (i, j)$ be the edge corresponding to an integer entry $c = C_{ij}$ which is neither 0 nor 1. Let $\Gamma = (\Gamma, s, t)$ be a binary algebraic branching program with path value c and $\mathcal{O}(\log |c|)$ many vertices, which exists by Proposition 2.1.1.

We will now replace the edge (i, j) by Γ (see Figure 2.1.1): Let G be the digraph that arises from $H \cup \Gamma$ by removing the edge (i, j), adding edges (i, s) and (t, j) with label 1 and adding loops with label 1 to all vertices of Γ. The directed adjacency matrix of G has size $d + \mathcal{O}(\log |c|) \leq d + \mathcal{O}(\log |c_{\max}|)$ and contains $k - 1$ integer entries which are neither 0 nor 1. By applying the induction hypothesis to the directed adjacency matrix of G, we obtain a matrix A of size

$$d + \mathcal{O}(\log |c_{\max}|) + (k-1) \cdot \mathcal{O}(\log |c_{\max}|) = d + k \cdot \mathcal{O}(\log |c_{\max}|)$$

whose determinant equals the value of G. We are left to show that the value of G is equal to $\det(C)$, i.e., the value of H.

For this purpose, we will analyze the relation between cycle covers of G and H, which is straightforward (see Figure 2.1.1): Consider a cycle cover K of G. Any vertex of Γ which is not covered by its loop must be part of a cycle whose intersection with Γ is a path from s to t. To K we can therefore associate a cycle cover K^H of H as follows: If every vertex of Γ is covered by its loop in K, let K^H be K without these loops. Otherwise, there is a unique cycle κ_K in K that restricts to an s-t-path π_K in Γ. Let κ_K^H be the intersection $\kappa_K \cap H$ together with the edge (i,j) and note that κ_K^H is a cycle in H. We obtain K^H from K by replacing κ_K with κ_K^H and removing all remaining loops from inside Γ.

All cycle covers L of H are of the form $L = K^H$ for some cycle cover K of G. If L is a cycle cover of H containing the edge (i,j) then the cycle covers K of G with $L = K^H$ are in bijection with the s-t-paths in Γ. We now fix such a cycle cover L. By definition of the value of a digraph, it suffices to show that

$$\sum_{\substack{K \text{ cycle cover of } G\\ \text{such that } L=K^H}} \operatorname{wt}(K) = \operatorname{wt}(L).$$

Note that K and $L = K^H$ differ only in loops and in the cycles κ_K and κ_K^H, respectively. Since loops contribute a factor of 1 to the weight of a cycle cover, we are left to prove that

$$\sum_{\substack{K \text{ cycle cover of } G\\ \text{such that } L=K^H}} \operatorname{wt}(\kappa_K) = \operatorname{wt}(\kappa_K^H).$$

Let $e_1, \ldots, e_r$ be the labels of the edges of $\kappa_K \cap H$. These are the edges shared by κ_K and κ_K^H. Thus,

$$\begin{aligned}
\operatorname{wt}(\kappa_K^H) &= (-1)^r \cdot c \cdot e_1 \cdots e_r = \left(\sum_{\substack{\pi \text{ is } s\text{-}t\text{-path}\\ \text{inside } P}} \operatorname{wt}(\pi) \right) \cdot (-1)^r \cdot e_1 \cdots e_r \\
&= \left(\sum_{\substack{K \text{ cycle cover of } G\\ \text{such that } L=K^H}} \operatorname{wt}(\pi_K) \right) \cdot (-1)^r \cdot e_1 \cdots e_r \\
&= \left(\sum_{\substack{K \text{ cycle cover of } G\\ \text{such that } L=K^H}} \operatorname{wt}(\pi_K) \cdot (-1)^r \cdot e_1 \cdots e_r \right) = \sum_{\substack{K \text{ cycle cover of } G\\ \text{such that } L=K^H}} \operatorname{wt}(\kappa_K)
\end{aligned}$$

is precisely the desired equality. ∎

2.1.3 Corollary. For every polynomial $P \in \mathbb{Z}[\mathbf{x}]$ there exists a binary variable matrix whose determinant is P.

Proof. Combine Theorem 1.5.4 and Proposition 2.1.2. ∎

2.2 Lower Bounds

This section is dedicated to the proof of Theorem 2.0.2. Let $\mathbb{B} := \{0, 1\}$. A sequential numbering makes the proof much easier to read, so we think of the variables as arranged in a 3×3 matrix

$$x = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}.$$

In this section, we will understand $\mathrm{per}_3 = \mathrm{per}(x)$ as a polynomial in the variables $x_1, \ldots, x_9$ instead of the variables x_{ij} with $1 \leq i, j \leq 3$.

2.2.1 Proof Outline

Let $d \in \mathbb{N}$ and A an $d \times d$ binary variable matrix. The binary matrix $B(A) \in \mathbb{B}^{d \times d}$ is defined as the matrix arising from A by setting all variables to 1. We call $B(A)$ the **support matrix** of A. If we set all variables to 1 in per_3, we obtain the value 6, so if $\mathrm{per}_3 = \det(A)$, then substituting 1 for all variables on both sides of the equation, we obtain the condition

$$6 = \det(B(A)). \tag{1}$$

In [EZ62; Slo11], the maximal values of determinants of binary matrices are computed for small values of d. Since

$$\forall B \in \mathbb{B}^{5 \times 5}\colon\ \det(B) \leq 5, \tag{2}$$

we immediately obtain the lower bound $\mathrm{bdc}(\mathrm{per}_3) \geq 6$.

Unfortunately, there are several matrices $B \in \mathbb{B}^{6 \times 6}$ that satisfy $\det(B) = 6$. We proceed in two steps to verify that nevertheless, none of these matrices B is the support matrix $B(A)$ of a candidate matrix A with $\mathrm{per}_3 = \det(A)$. A rough outline is the following:

(a) Enumerate all matrices $B \in \mathbb{B}^{6 \times 6}$ with $\det(B) = 6$ up to symmetries.

(b) For all those matrices B prove that B is not the support matrix $B(A)$ of a binary variable matrix A with $\det(A) = \mathrm{per}_3$. We describe this process in the next subsection.

2.2.2 Stepwise Reconstruction

Let us make (b) precise. In the hope of failing, we attempt to reconstruct a binary variable matrix A that has support B and which also satisfies $\det(A) = \mathrm{per}_3$. During the reconstruction process, we successively replace 1's in B by the next variable. The process is as follows:

Given a binary matrix $B \in \mathbb{B}^{6\times 6}$, let

$$S := \{(i,j) \mid B_{ij} = 1\}$$

be the set of possible variable positions. For any set of positions $I \subseteq S$, we consider the matrix B_I that arises from B by placing a variable y in every position in I. If B is the support of a binary variable matrix A with $\det(A) = \mathrm{per}_3$ and I contains exactly the positions where $y := x_1$ occurs in A, then $\det(B_I)$ must be equal to

$$\mathrm{per}_3 \begin{pmatrix} y & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 2y + 4. \tag{3}$$

We define the set

$$\mathcal{S} := \{I \subseteq S \mid \det(B_I) = 2y + 4\}.$$

2.2.1 Lemma. Let A be a binary variable matrix with support B and $\det(A) = \mathrm{per}_3$. Let $k \in \{1, \ldots, 9\}$ and define $I_k := \{(i,j) \mid A_{ij} = x_k\}$ as the set of positions where the variable x_k occurs in A. Then, we have $I_k \in \mathcal{S}$.

Proof. By the symmetry of the permanent, we may assume that $k = 1$. In the matrix A, setting every variable except $y := x_1$ to 1 yields the matrix B_I and therefore, $\det(B_I) = 2y + 4$ as in (3), because $\det(A) = \mathrm{per}_3$. This means $I_k \in \mathcal{S}$ by definition. ■

Therefore, if B is the support matrix $B(A)$ of a binary variable matrix A with $\det(A) = \mathrm{per}_3$, we can find 9 pairwise disjoint sets in $\mathcal{S}$, one for each variable x_k, that specify precisely where to place these variables in A. By a recursive search and backtracking, we now look for sets $I_1, \ldots, I_k \in \mathcal{S}$ such that

- i $I_1, \ldots, I_k$ are pairwise disjoint.
- ii Placing x_i into B at every position from I_i for $1 \leq i \leq k$ yields a matrix A_k such that $\det(A_k)$ is equal to $\mathrm{per}_3(x_1, \ldots, x_k, 1, \ldots, 1)$.

The search is recursive in the following sense: First, the possible choices at depth $k = 1$ are given by $\mathcal{S}$. Enumerating the possible choices for depth $k + 1$ works as follows: For each choice $I_1, \ldots, I_k \in \mathcal{S}$ with the above two properties, we enumerate all $I_{k+1} \in \mathcal{S}$ that have empty intersection with $I_1 \cup \cdots \cup I_k$ and check whether condition (ii) is satisfied.

If the recursive search never reaches $k = 9$ or fails there, then B is not the support of a binary variable matrix A with $\det(A) = \mathrm{per}_3$. If we reach level 9 however and do not fail there, we have found such an A.

In practice, the evaluation of $\det(A)$ is sped up significantly by working over a large finite field $\mathbb{F}_p$ and choosing random elements $x_1, \ldots, x_9 \in \mathbb{F}_p \setminus \{0, 1\}$.

2.2.3 Exploiting Symmetries in Enumeration

Let us call two matrices *equivalent* if they arise from each other by transposition and/or permutation of rows and/or columns. A key observation is that equivalent matrices have the same determinant up to sign. Therefore we do not have to list all binary matrices $B \in \mathbb{B}^{6\times 6}$ with $\det(B) = 6$, but it suffices to list one representative matrix B with $\det(B) = \pm 6$ for each equivalence class. It happens to be the case that the equivalence classes of 6×6 binary matrices are in bijection to graph isomorphy classes of undirected bipartite graphs $G = (V \cup W, E)$ with $|V| = |W| = 6$, $V \cap W = \emptyset$ as follows: For $V = \{v_1, \ldots, v_6\}$ and $W = \{w_1, \ldots, w_6\}$, the bipartite adjacency matrix $B(G) \in \mathbb{B}^{6\times 6}$ of G is defined via $B(G)_{i,j} = 1$ if and only if $\{v_i, w_j\} \in E$. Row and column permutations in $B(G)$ are reflected by renaming vertices in G. Transposition of $B(G)$ amounts to switching V and W in G.

The computer software nauty [MP13] can enumerate all 251 610 of these bipartite graphs, which is already a significant improvement over the $2^{36} = 68\,719\,476\,736$ elements of $\mathbb{B}^{6\times 6}$. To further limit the number of bipartite graphs that have to be considered, we make the following observations:

- We need not consider binary matrices B containing a row i with only a single entry B_{ij} equal to 1. Indeed, Laplace expansion over the i-th row yields that $\det(B)$ is equal to the determinant of a 5×5 binary matrix, which can at most be 5, see (2). Translating to bipartite graphs, we only need to consider those bipartite graphs where all vertices have at least two neighbours.
- If two distinct vertices in G have the same neighbourhood, then the bipartite adjacency matrix $B(G)$ has two identical rows (or columns) which would imply $\det(B(G)) = 0$. Hence, we only need to enumerate bipartite graphs where all vertices have distinct neighbourhoods. Unfortunately nauty can impose this restriction only on rows and not on columns.

With these restrictions, the nauty command

```
genbg -d2:2 -z 6 6
```

generates 44 384 bipartite graphs, only 263 of which have a bipartite adjacency matrix with determinant equal to ± 6. We then preprocess this list by swapping the first two rows of any matrix with negative determinant.

Finally, the stepwise reconstruction (Subsection 2.2.2) fails for all of these 263 matrices, proving that $\mathrm{bdc}(\mathrm{per}_3) \geq 7$. The algorithm takes 28 seconds on an Intel Core™ i7-4500U CPU (2.4 GHz) to finish.

Unfortunately, $\mathrm{bdc}(\mathrm{per}_4)$ can currently not be determined in this fashion because the enumeration of all apropriate bipartite graphs, already on $9+9$ vertices, is infeasible.

2.3 Uniqueness of Grenet's construction in the 7×7 case

The methods from Section 2.2 can be used to determine all 7×7 binary variable matrices A with the property that $\det(A) = \mathrm{per}_3$. By means of a cluster computation over the course of one week, we determined all 463 binary variable matrices with this property and made some noteworthy discoveries.

The Grenet construction (see Section 2.5) yields the matrix

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & x_{32} & x_{33} & 0 & 0 \\ 0 & 1 & 0 & x_{31} & 0 & x_{33} & 0 \\ 0 & 0 & 1 & 0 & x_{31} & x_{32} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & x_{23} \\ 0 & 0 & 0 & 0 & 1 & 0 & x_{22} \\ 0 & 0 & 0 & 0 & 0 & 1 & x_{21} \end{pmatrix}. \tag{4}$$

It is the unique "sparse" 7×7 binary variable matrix from among the 463, in the sense that every other matrix from the list has more than three nonzero entries in some row or column.

Motivated by the above observation, we verified by hand (with computer support) that in fact, all of the 463 matrices can be reduced to (4) by means of elementary row and column operations. This can be summarized as follows:

2.3.1 Proposition. Every 7×7 binary variable matrix A with $\det(A) = \mathrm{per}_3$ is equivalent to the Grenet construction (4) under the following two group actions:

(1) The action of $\{(g,h) \mid \det(g) = \det(h)\} \subseteq \mathrm{GL}_7(\mathbb{Z}) \times \mathrm{GL}_7(\mathbb{Z})$ on 7×7 matrices via left and right multiplication, together with transposition of 7×7 matrices.

(2) The action of $\mathfrak{S}_3 \times \mathfrak{S}_3$ on the variables x_{ij} with $1 \leq i,j \leq 3$, and the corresponding transposition (i.e., the map $x_{ij} \mapsto x_{ji}$.)

Note that (1) leaves the determinant of any 7×7 binary variable matrix invariant and (2) leaves the permanent polynomial invariant. ■

2.3.2 Example. One of the matrices that occur in our enumeration is the matrix

$$A := \begin{pmatrix} x_{31} & x_{32} & x_{31} & 0 & x_{32} & 1 & x_{23} \\ 1 & x_{33} & 0 & x_{31} & x_{33} & x_{31} & x_{22} \\ x_{33} & 0 & x_{33} & x_{32} & 1 & x_{32} & x_{21} \\ 1 & 0 & 1 & 0 & 0 & 0 & x_{22} \\ 0 & x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & x_{21} \\ 0 & 0 & 0 & 1 & 0 & 1 & x_{23} \end{pmatrix}.$$

One can check that indeed $\det(A) = \mathrm{per}_3$. In this case, the matrices

$$g := \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad h := \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

are both invertible over $\mathbb{Z}$ and gAh is precisely (4). ●

2.4 Algebraic Complexity Classes

In this section we relate binary determinantal complexity to classical complexity measures. An algebraic circuit is called **skew** if for every multiplication gate at least one of its two parents is an input gate. We denote by $\mathrm{cc_s}(P)$ the minimum number d such that P can be computed by a skew circuit of size d. $\mathbf{VP_s}$ consists of all p-families P such that $\mathrm{cc_s}(P_m) \in \mathrm{poly}(m)$.

If all input gates of a circuit C are labeled with either 1, -1, or a variable, the circuit is called **constant-free**. Note that every constant-free circuit computes a polynomial that has integer coefficients. We denote by cc^0_s and $\mathrm{cc}^0_\mathrm{ws}$ analogously defined complexity measures with the additional condition that only constant-free circuits are allowed. The complexity classes $\mathbf{VP}^0_\mathbf{s}$ and $\mathbf{VP}^0_\mathbf{ws}$ are defined as consisting of all p-families P such that $\mathrm{cc}^0_\mathrm{s}(P_m)$ and $\mathrm{cc}^0_\mathrm{ws}(P_m)$ are respectively sequences in $\mathrm{poly}(m)$. See also [Mal03].

A fundamental result in [Tod92] (see also [MP08]) is that $\mathbf{VP_{ws}} = \mathbf{VP_s}$, so $\mathbf{VP_s}$ is another class for which the determinant is complete – recall Subsection 1.5.1. Analyzing the constants which appear in the proof of $\mathbf{VP_{ws}} = \mathbf{VP_s}$ in [Tod92], we see that the proof immediately yields $\mathbf{VP}^0_\mathbf{ws} = \mathbf{VP}^0_\mathbf{s}$. For the sake of comparison with $\mathbf{VP}^0_\mathbf{s}$, let us make the following definition.

2.4.1 Definition. The complexity class $\mathbf{DETP}^0$ consists of all sequences of polynomials that have polynomially bounded binary determinantal complexity bdc. ●

The main purpose of this section is to show the following statement.

2.4.2 Proposition. $\mathbf{VP}^0_{\mathbf{ws}} = \mathbf{VP}^0_{\mathbf{s}} = \mathbf{DETP}^0$.

Proof. The proof of [Tod92, Lemma 3.4] immediately shows that $\mathbf{DETP}^0 \subseteq \mathbf{VP}^0_{\mathbf{s}}$. To show that $\mathbf{VP}^0_{\mathbf{s}} \subseteq \mathbf{DETP}^0$ we adapt the proof of [Tod92, Lemma 3.5 or Theorem 4.3], but a subtlety arises: The proof shows that from a weakly skew or skew circuit C we can construct a matrix A' of size polynomially bounded in the number of vertices in C such that $\det(A')$ is the polynomial computed by C with the drawback that A' is not a binary variable matrix, but A' has as entries variables and constants 0, 1, and -1. Fortunately Proposition 2.1.2 establishes $\mathbf{DETP}^0 = \mathbf{VP}^0_{\mathbf{s}} = \mathbf{VP}^0_{\mathbf{ws}}$. ■

2.4.3 Remark. In the past, other models of computation with bounded coefficients have already given way to stronger lower bounds than their corresponding unrestricted models: [Mor73] on the Fast Fourier Transform, [Raz03] on matrix multiplication, and [BL04] on arithmetic operations on polynomials. ●

From Valiant's completeness result [Val79a] we deduce that $\mathbf{VP} \neq \mathbf{VNP}$ implies $\mathrm{per}_m \notin \mathbf{VP}^0_{\mathbf{ws}}$. A main goal is to prove $\mathrm{per}_m \notin \mathbf{VP}^0_{\mathbf{ws}}$ unconditionally. This could be a simpler question than $\mathbf{VP} \neq \mathbf{VNP}$ or even $\mathbf{VP}^0 \neq \mathbf{VNP}^0$, because with what is known today, from $\mathrm{per}_m \in \mathbf{VP}^0_{\mathbf{ws}}$ we cannot conclude $\mathbf{VP}^0 = \mathbf{VNP}^0$, see [Koi04, Thm. 4.3]. If we replace the permanent polynomial by the Hamiltonian Cycle polynomial

$$\mathrm{HC}_m := \sum_{\substack{\pi \in \mathfrak{S}_m \\ \pi \text{ is } m\text{-cycle}}} \prod_{i=1}^{m} x_{i,\pi(i)},$$

then the question $\mathrm{HC}_m \notin \mathbf{VP}_{\mathbf{ws}}$ is indeed equivalent to separating $\mathbf{VP}^0_{\mathbf{ws}}$ from $\mathbf{VNP}^0$, see [Koi04, Thm. 2.5], mutatis mutandis.

We ran our analysis for HC_m, $m \leq 4$ and proved $\mathrm{bdc}(\mathrm{HC}_1) = 1$, $\mathrm{bdc}(\mathrm{HC}_2) = 2$, $\mathrm{bdc}(\mathrm{HC}_3) = 3$, $\mathrm{bdc}(\mathrm{HC}_4) \geq 7$. The matrices are given at the end of Section 2.5 in (5). This means that $7 \leq \mathrm{bdc}(\mathrm{HC}_4) \leq 13$, where the upper bound follows from considerations analogous to Grenet's construction, see Section 2.5.

2.5 Graph Constructions for Polynomials

In this section, we review the proof of Theorem 2.0.1 from [Gre11]. Furthermore, we use the same methods to prove the following result about the Hamiltonian Cycle polynomial:

2.5.1 Theorem. We have $\mathrm{bdc}(\mathrm{HC}_{m+1}) \leq m \cdot 2^{m-1} + 1$ for all $m \in \mathbb{N}$.

Recall that we denote by $[m] := \{1, \ldots, m\}$ the set of numbers between 1 and m.

2.5.1 Grenet's Construction for the Permanent

We prove Theorem 2.0.1. The construction of Grenet is a digraph Γ whose vertices $V := \{v_I \mid I \subseteq [m]\}$ are indexed by the subsets of $[m]$. Hence, $d := |V| = 2^m$. We partition $V = V_0 \cup \cdots \cup V_m$ such that V_i contains the vertices belonging to subsets of size i. We set $s := v_\emptyset$ and $t := v_{[m]}$, so $V_0 = \{s\}$ and $V_m = \{t\}$. Edges will go exclusively from V_{i-1} to V_i for $i \in [m]$. In fact, we insert an edge from v_I to v_J if and only if there is some $j \in [m]$ with $J = I \cup \{j\}$. This edge is then labeled with the variable x_{ij}, where $i = |J|$. For example, there are m edges going from V_0 to V_1, one for each variable x_{1j} with $1 \leq j \leq m$. It is clear that for each permutation $\pi \in \mathfrak{S}_m$, there is precisely one s-t-path in Γ whose path weight is $(-1)^{m-1} \cdot x_{1,\pi(1)} \cdots x_{m,\pi(m)}$. Consequently, the path value of the algebraic branching program $\Gamma = (\Gamma, s, t)$ is equal to $(-1)^{m-1} \cdot \mathrm{per}_m$. Theorem 2.0.1 then follows from the following lemma:

2.5.2 Lemma. Let $\Gamma = (\Gamma, s, t)$ be a binary algebraic branching program on $d \geq 3$ vertices with path value $\pm P$. Then, there is a binary variable matrix of size $d-1$ whose determinant is equal to P.

Remark. The proof of this lemma is essentially identical to the proof of Theorem 1.5.4, but it seems convenient to present it anyway. ○

Proof. We first construct a graph G from Γ by identifying the two vertices s and t and adding loops with label 1 to every other vertex. The s-t-paths in Γ are then in one-to-one correspondence with the cycle covers of G: Indeed, any cycle cover in G must cover the vertex $s = t$ and this cycle corresponds to an s-t-path in Γ. Every other vertex can only be covered by its loop because Γ is acyclic. The graph G now has the value $\pm P$ by definition and its directed adjacency matrix A has size $d-1$. Since $d - 1 \geq 2$, we can exchange the first two rows of A to change the sign of its determinant. ■

2.5.2 Hamiltonian Cycle Polynomial

In this subsection, we prove Theorem 2.5.1 using Lemma 2.5.2. In order to construct a binary algebraic branching program $\Gamma = (\Gamma, s, t)$ with path value HC_{m+1}, we proceed similar to Grenet's construction for the permanent. We will refer to cyclic permutations in $\mathfrak{S}_{m+1}$ of order $m+1$ simply as *cycles* because no cyclic permutations of lower order will be considered. Observe that the cycles in $\mathfrak{S}_{m+1}$ are in bijection with the permutations in $\mathfrak{S}_m$. This can be seen by associating to $\pi \in \mathfrak{S}_m$ the cycle $\sigma = (\pi(1), \ldots, \pi(m), m+1) \in \mathfrak{S}_{m+1}$. In other words, σ maps $m+1$ to $\pi(1)$, it maps $\pi(1)$ to $\pi(2)$ and so on.

In addition to two vertices s and t, our binary algebraic branching program will have a vertex $v_{(I,i)}$ for every nonempty subset $I \subseteq [m]$ and $i \in I$. By our above Lemma 2.5.2, the resulting binary variable matrix will have a size of

$$1 + \sum_{i=1}^{m} \binom{m}{i} \cdot i = m \cdot 2^{m-1} + 1.$$

For $m = 3$, this is equal to $3 \cdot 2^2 + 1 = 13$.

We construct the edges in Γ so that every cycle $\sigma = (a_1, \ldots, a_m, m+1)$ corresponds to an s-t-path which has $v_{(I,i)}$ as its k-th vertex if and only if $I = \{a_1, \ldots, a_k\}$ and $i = a_k$. We insert the following edges:

- from s to $v_{(\{i\},i)}$ for each $i \in [m]$ with label $x_{m+1,i}$
- from $v_{(I,i)}$ to $v_{(I \cup \{j\},j)}$ for each $i \in I \subseteq [m]$ and $j \in [m] \setminus I$ with label $x_{i,j}$
- from $v_{([m],i)}$ to t for each $i \in [m]$ with label $x_{i,m+1}$.

We can again partition the set of vertices as $V = V_0 \cup \ldots \cup V_{m+1}$ where $V_0 = \{s\}$, $V_{m+1} = \{t\}$ and for $k \in [m]$, the set V_k consists of all vertices $v_{(I,i)}$ with $|I| = k$. Then, edges go only from V_k to V_{k+1}, in particular Γ is acyclic. Furthermore, all s-t-paths in Γ have the same lengths and correspond uniquely to cycles in $\mathfrak{S}_{m+1}$. This concludes the proof of Theorem 2.5.1.

We know of no better construction for arbitrary m, but for small m we have

$$\mathrm{HC}_2 = \det \begin{pmatrix} x_{12} & 0 \\ 0 & x_{21} \end{pmatrix} \qquad \mathrm{HC}_3 = \det \begin{pmatrix} 0 & x_{12} & x_{13} \\ x_{21} & 0 & x_{23} \\ x_{31} & x_{32} & 0 \end{pmatrix}. \tag{5}$$

Chapter 3

Geometric Complexity Theory

Recall Definition 1.3.4 for the notation $P \leq_R Q$ when $P, Q \in R[\mathbf{x}]$ are polynomials over a commutative ring R. We call two polynomials P and Q **equivalent** if $P \leq_R Q$ and $Q \leq_R P$. We write $P \simeq_R Q$ in this case and $P \simeq Q$ if there is no ambiguity concerning the ring R. This notion has been used for p-families already [Bü00, p. 8] but we require it here for single polynomials only.

It follows directly from the definition that circuit complexity and (binary) determinantal complexity are examples for complexity measures in the following sense:

3.0.1 Definition. Let R be a commutative ring. An **arithmetic complexity measure** over R is a function $c\colon R[\mathbf{x}] \to \mathbb{N}$ such that $P \leq_R Q$ implies $c(P) \leq c(Q)$. ●

Remark. For every complexity measure c, equivalent polynomials clearly have the same complexity with respect to c. Observe also that equivalent polynomials are identical up to renaming the variables, which follows easily from the definition. It is straightforward to check that "$\simeq$" is an equivalence relation. ○

Notation. If $\mathcal{P}$ and $\mathcal{Q}$ are two sets of polynomials, we write $\mathcal{P} \simeq \mathcal{Q}$ to indicate that $(\mathcal{P}/\simeq) = (\mathcal{Q}/\simeq)$, i.e., the equivalence classes of polynomials represented by $\mathcal{P}$ and $\mathcal{Q}$ are the same. ○

We will work exclusively over the field $R = \mathbb{C}$ of complex numbers from here on in, mainly because it simplifies the representation theory and algebraic geometry that enters the picture later.

3.1 Orbit and Orbit Closure as Complexity Measure

We will now introduce a new complexity measure which is the first step towards a geometric interpretation of Conjecture 1.5.3. Let $W_d := \mathbb{C}^{d\times d}$ be the space of $d \times d$ square matrices. It is a complex vector space and $\mathrm{End}(W_d)$ denotes the set of all endomorphisms $a\colon W_d \to W_d$. The determinant is a map $\det_d\colon W_d \to \mathbb{C}$, so for $a \in \mathrm{End}(W_d)$, we can consider the composition $\det_d \circ a$, which is again a polynomial function: This is just linear substitution of variables. We then define the **determinantal orbit complexity** of P as

$$\operatorname{doc}(P) := \min\{d \in \mathbb{N} \mid \exists a \in \operatorname{End}(W_d) \colon P \leq \det_d \circ a\}\,, \tag{1}$$

which is well-defined because $\operatorname{doc}(P) \leq \operatorname{dc}(P)$ for all polynomials P. Our main goal will be to show that the two measures are actually equivalent in the following sense:

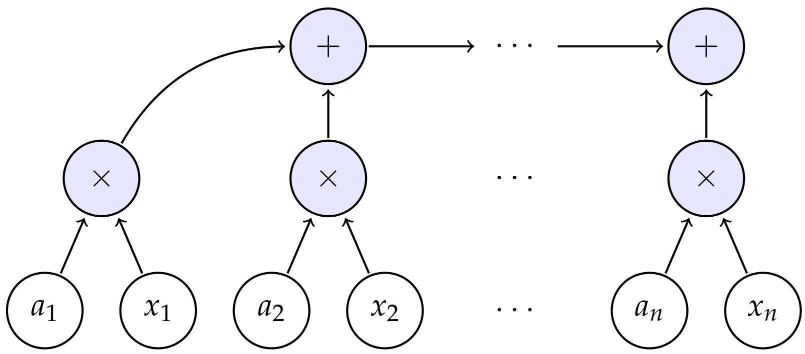
3.1.1 Definition. Let X be a set and $c_1, c_2 \colon X \to \mathbb{N}$ two functions. We say that c_1 and c_2 are **polynomially equivalent** if there exist $t_1, t_2 \in \text{poly}$ such that $c_1(x) \leq t_2(c_2(x))$ and $c_2(x) \leq t_1(c_1(x))$ for all $x \in X$. We denote this by $c_1 \equiv c_2$.

3.1.2 Proposition. We have $\operatorname{dc} \equiv \operatorname{doc}$.

Proof. The inequality $\operatorname{doc} \leq \operatorname{dc}$ is clear. The following Lemma 3.1.3 states that there is a function $t \in \text{poly}$ such that $\operatorname{dc}(\det_d \circ a) \leq t(d)$ for every $a \in \operatorname{End}(\mathbb{C}^{d\times d})$. Since any polynomial P with $d := \operatorname{doc}(P)$ admits a linear transformation $a \in \operatorname{End}(\mathbb{C}^{d\times d})$ with $P \leq \det_d \circ a$, we have $\operatorname{dc}(P) \leq \operatorname{dc}(\det_d \circ a) \leq t(d) = t(\operatorname{doc}(P))$. ∎

3.1.3 Lemma. There exists a function $t \in \text{poly}$ such that for all $d \in \mathbb{N}$ and all linear maps $a \colon \mathbb{C}^{d\times d} \to \mathbb{C}^{d\times d}$, we have $\operatorname{dc}(\det_d \circ a) \leq t(d)$.

Proof. There is some $s \in \text{poly}$ and weakly skew circuits $\tilde{C}_d$ computing the polynomial $\det_d$ with $|\tilde{C}_d| \leq s(d)$, see [MV97]. Let now $a \colon \mathbb{C}^{d\times d} \to \mathbb{C}^{d\times d}$ be a linear map and denote by $a_{ij} \colon \mathbb{C}^{d\times d} \to \mathbb{C}$ its (i,j)-th component. Each a_{ij} is a linear form in $n = d^2$ many variables. Note that a linear form $\sum_{i=1}^{n} a_i x_i$ with $a_1, \ldots, a_n \in \mathbb{C}$ can be computed by a weakly skew circuit of size $2n - 1$:



(2)

We construct a weakly skew circuit C_d computing $\det_d \circ a$ as follows: Take the disjoint union of $\tilde{C}_d$ and d^2 circuits of the form (2), each of which respectively computes a_{ij} for all $1 \leq i, j \leq d$. Then, identify the input gate labeled x_{ij} in $\tilde{C}_d$ with the gate computing a_{ij}. The result is a weakly skew circuit C which computes $\det_d \circ a$ and $|C| < |\tilde{C}_d| + 2d^4$. Hence Theorem 1.5.2 implies

$$\operatorname{dc}(\det_d \circ a) \leq |C| + 1 \leq |\tilde{C}_d| + 2d^4 \leq s(d) + 2d^4 = t(d).$$

∎

3.2 Border Complexity

Let $W_d := \mathbb{C}^{d \times d}$ be the space of $d \times d$ square matrices. By Proposition 3.1.2, Conjecture 1.5.3 can be expressed in terms of the values $d, m \in \mathbb{N}$ where per_m is a projection of some polynomial in $\det_d \circ \operatorname{End}(W_d)$.

3.2.1 Remark. Unfortunately, $\det_d \circ \operatorname{End}(W_d)$ is unattractive from a geometric point of view because it is neither a closed nor an open set for the Euclidean or the Zariski topology in general. We will now replace this set by its closure, which corresponds to allowing arbitrary approximation in our computational model. This is arguably natural from both a computational and a geometric point of view, but the implications for the computational model are not completely understood yet. ●

For $\mathbf{y} = \{y_1, \ldots, y_n\} \subseteq \mathbf{x}$ and $d \in \mathbb{N}$, the space $\mathbb{C}[\mathbf{y}]_{\leq d} = \{P \in \mathbb{C}[\mathbf{y}] \mid \deg(P) \leq d\}$ is a finite-dimensional vector space which we endow with the Euclidean topology and we consider on $\mathbb{C}[\mathbf{y}]$ the final topology with respect to the inclusions $\mathbb{C}[\mathbf{y}]_{\leq d} \subseteq \mathbb{C}[\mathbf{y}]$. Lastly, the topology on $\mathbb{C}[\mathbf{x}]$ that we use is the final topology with respect to the inclusions $\mathbb{C}[\mathbf{y}] \subseteq \mathbb{C}[\mathbf{x}]$ for all finite subsets $\mathbf{y} \subseteq \mathbf{x}$.

For any complexity measure $c\colon \mathbb{C}[\mathbf{x}] \to \mathbb{N}$, we define the corresponding **border complexity** function as

$$\underline{c}(P) := \min\left\{ d \in \mathbb{N} \,\middle|\, P \in \overline{c^{-1}([d])} \right\},$$

where we use the notation $[d] := \{1, \ldots, d\}$ and hence, $c^{-1}([d]) = \{P \mid c(P) \leq d\}$. If c measures complexity, then $\underline{c}$ measures approximate complexity: $\underline{c}(P) \leq d$ means that any neighbourhood of P contains a polynomial of complexity at most d. Clearly, $\underline{c}(P) \leq c(P)$ always holds.

This defines the **determinantal orbit border complexity** $\underline{\mathrm{doc}}\colon \mathbb{C}[\mathbf{x}] \to \mathbb{N}$ and the **determinantal border complexity** $\underline{\mathrm{dc}}\colon \mathbb{C}[\mathbf{x}] \to \mathbb{N}$. We note that $\underline{\mathrm{dc}}$ and $\underline{\mathrm{doc}}$ are polynomially equivalent by general principle:

3.2.2 Proposition. For two complexity measures c_1 and c_2, $c_1 \equiv c_2$ implies $\underline{c_1} \equiv \underline{c_2}$.

Proof. There is a $t \in \mathrm{poly}$ such that $c_1(P) \leq t(c_2(P))$ for all $P \in \mathbb{C}[\mathbf{x}]$. In other words, we have $c_2^{-1}([d]) \subseteq c_1^{-1}([t(d)])$. If $P \in \mathbb{C}[\mathbf{x}]$ satisfies $d := \underline{c_2}(P)$, then it follows from an elementary topological argument that $P \in \overline{c_2^{-1}([d])} \subseteq \overline{c_1^{-1}([t(d)])}$, hence $\underline{c_1}(P) \leq t(d) = t(\underline{c_2}(P))$. The statement follows by symmetry. ■

With Proposition 3.1.2 we obtain:

3.2.3 Corollary. $\underline{\mathrm{doc}} \equiv \underline{\mathrm{dc}}$. ■

The precise statement of the conjecture by Mulmuley and Sohoni is the following:

3.2.4 Conjecture ([MS01, 4.3]). $\underline{\mathrm{doc}}(\mathrm{per}_m)$ is not polynomially bounded in m.

While the converse is not known, Conjecture 3.2.4 implies Conjecture 1.5.3 because we have $\underline{\mathrm{doc}}(\mathrm{per}_m) \leq \mathrm{doc}(\mathrm{per}_m)$: If $\underline{\mathrm{doc}}(\mathrm{per}_m) \notin \mathrm{poly}(m)$, then we also have $\mathrm{doc}(\mathrm{per}_m) \notin \mathrm{poly}(m)$. For homogeneous polynomials such as the permanent, we can give a more concrete description of $\underline{\mathrm{doc}}$. We denote by $\mathrm{GL}(W_d) \subseteq \mathrm{End}(W_d)$ the set of invertible endomorphisms, the **general linear group** on W_d.

3.2.5 Proposition. For a homogeneous polynomial $P \in \mathbb{C}[\mathbf{x}]_m$ and any $x \in \mathbf{x}$ with $x \notin \mathrm{supp}(P)$, we have

$$\underline{\mathrm{doc}}(P) = \min\left\{ d \in \mathbb{N} \;\middle|\; \exists Q \in \overline{\det_d \circ \mathrm{GL}(W_d)} \colon x^{d-m}P \simeq Q \right\}.$$

For the proof, we require the following observations:

3.2.6 Lemma. We have $\overline{\det_d \circ \mathrm{End}(W_d)} = \overline{\det_d \circ \mathrm{GL}(W_d)}$.

Proof. We only have to show the inclusion "$\subseteq$". Since the right hand side is closed, it is in fact sufficient to show that it contains $\det_d \circ \mathrm{End}(W_d)$. Consider the map

$$\begin{aligned} \omega\colon \mathrm{End}(W_d) &\longrightarrow \mathbb{C}[\mathbf{x}] \\ a &\longmapsto \det_d \circ a \end{aligned}$$

It is continuous because the coefficients of $\det_d \circ a$ are polynomials in the entries of (a matrix representation of) a. Since $\mathrm{GL}(W_d)$ is dense in $\mathrm{End}(W_d)$, we have

$$\det_d \circ \mathrm{End}(W_d) = \omega(\mathrm{End}(W_d)) = \omega(\overline{\mathrm{GL}(W_d)}) \subseteq \overline{\omega(\mathrm{GL}(W_d))} = \overline{\det_d \circ \mathrm{GL}(W_d)}. \qquad \blacksquare$$

3.2.7 Lemma. For two sets $\mathcal{P}, \mathcal{Q} \subseteq \mathbb{C}[\mathbf{x}]$ of polynomials, $\mathcal{P} \simeq \mathcal{Q}$ implies $\overline{\mathcal{P}} \simeq \overline{\mathcal{Q}}$.

Proof. Since "$\simeq$" is an equivalence relation, we can give $\mathbb{C}[\mathbf{x}]/\simeq$ the quotient topology and we claim that the quotient map $\pi\colon \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}]/\simeq$ closed. Let $\mathbf{y} \subseteq \mathbf{x}$ be a finite subset. We denote by $\mathfrak{S}_{\mathbf{y}}$ the group of all set automorphisms of $\mathbf{y}$. For $P, Q \in \mathbb{C}[\mathbf{y}]$, we then have $P \simeq Q$ if and only if there is some $\pi \in \mathfrak{S}_{\mathbf{y}}$ with $P = Q^\pi$, in the sense of Definition 1.3.4. Hence, the equivalence class of any $P \in \mathbb{C}[\mathbf{y}]$ is finite, from which it follows easily that the quotient map $\mathbb{C}[\mathbf{y}] \to \mathbb{C}[\mathbf{y}]/\simeq$ is closed. Since $\mathbb{C}[\mathbf{x}]/\simeq$ has the final topology with respect to π and $\mathbb{C}[\mathbf{x}]$ has the final topology with respect to the inclusions $\mathbb{C}[\mathbf{y}] \subseteq \mathbb{C}[\mathbf{x}]$ for any finite $\mathbf{y} \subseteq \mathbf{x}$, it follows that π is closed. Therefore, $\pi(\mathcal{P}) = \pi(\mathcal{Q})$ implies $\pi(\overline{\mathcal{P}}) = \overline{\pi(\mathcal{P})} = \overline{\pi(\mathcal{Q})} = \pi(\overline{\mathcal{Q}})$. $\blacksquare$

Proof of Proposition 3.2.5. Let $x \in \mathbf{x}$ and $\tilde{\mathbf{x}} := \mathbf{x} \setminus \{x\}$. Note that every polynomial is equivalent to a polynomial in $\mathbb{C}[\tilde{\mathbf{x}}]$. The map $h_d\colon \mathbb{C}[\tilde{\mathbf{x}}]_m \to \mathbb{C}[\mathbf{x}]_d$ given by $P \mapsto x^{d-m}P$ is linear, in particular it is continuous.

Claim. If $\mathrm{doc}(P) \le d$, then there is some $a \in \mathrm{End}(W_d)$ with $h_d(P) \simeq \det_d \circ a$.

We will prove this claim later and show first that it implies the statement. Let doc_m be the restriction of doc to $\mathbb{C}[\mathbf{x}]_m$ and observe

$$\begin{aligned}
\overline{\mathrm{doc}_m^{-1}([d])} &= \overline{\{P \in \mathbb{C}[\mathbf{x}]_m \mid \mathrm{doc}(P) \le d\}} \\
&= \overline{\{P \in \mathbb{C}[\mathbf{x}]_m \mid \exists a \in \mathrm{End}(W_d) \colon P \le \det_d \circ a\}} && \text{(by the claim)} \\
&\simeq \overline{\{P \in \mathbb{C}[\tilde{\mathbf{x}}]_m \mid \exists a \in \mathrm{End}(W_d) \colon h_d(P) = \det_d \circ a\}} && \text{(by Lemma 3.2.7)} \\
&= \overline{h_d^{-1}(\det_d \circ \mathrm{End}(W_d))} && \text{(by definition)} \\
&= h_d^{-1}(\overline{\det_d \circ \mathrm{End}(W_d)}) && (h_d \text{ is continuous}) \\
&= h_d^{-1}(\overline{\det_d \circ \mathrm{GL}(W_d)}). && \text{(by Lemma 3.2.6)}
\end{aligned}$$

We are left to prove our claim. Assume that $\mathrm{doc}(P) \le d$, so there is an $a \in \mathrm{End}(W_d)$ with $P \le \det_d \circ a$. Potentially replacing P by an equivalent polynomial, we may assume that $\mathrm{supp}(P) \subseteq \mathrm{supp}(\det_d)$. Let $S \subseteq \mathrm{supp}(\det_d)$ and $\sigma \colon S \to S \cup \mathbb{C}$ be the map with $P = (\det_d \circ a)^\sigma$. After composing a with an appropriate linear transformation, we can assume that $\sigma \colon S \to \mathbb{C}^\times$, since mapping a variable to another variable or to zero is a linear operation on the parameter space. Since scaling the variables by a nonzero complex number is also a linear operation, we achieve that $\sigma \colon S \to \{1\}$. If $S = \emptyset$, then $P = \det_d \circ a$ and in particular $m = \deg(P) = d$ so we are done. Otherwise, we can compose a with a linear projection mapping all variables in S to one $x \in S$ and consequently replace S by $\{x\}$. Since P and $\det_d \circ a$ are both homogeneous, it follows that $x^{d-m}P = \det_d \circ a$. ■

3.3 The Flip via Obstructions

In this section, we freely use concepts and results from representation theory and geometric invariant theory, see Chapter A. We also expect the reader to be familiar with elementary topology.

In light of Proposition 3.2.5 and Conjecture 3.2.4, for $m \le d$, we define the **padded permanent** $\mathrm{pp}_{d,m} := x_{d,d}^{d-m} \cdot \mathrm{per}_m$ which is a degree d form on the space $W = \mathbb{C}^{d \times d}$.

3.3.1 Remark. The padding arises naturally from a computational perspective, but it causes significant problems from a geometric point of view. We will point out these problems when they occur and conclude the discussion in Chapter 4. ●

3.3.2 Definition. For $P \in \mathbb{C}[W]_d$, we define $\Omega(P) := P \circ \mathrm{GL}(W)$ the **orbit** of P. We denote by $\overline{\Omega}(P)$ the Euclidean closure of $\Omega(P)$. We also write Ω_P and $\overline{\Omega}_P$ instead of $\Omega(P)$ and $\overline{\Omega}(P)$, depending on which is more readable. ●

Let W be a finite-dimensional complex vector space and $P, Q \in \mathbb{C}[W]_d$ two homogeneous forms of degree d on W. In the context of Conjecture 3.2.4, they play the roles of $Q = \mathrm{pp}_{d,m}$ and $P = \det_d$ on $W = \mathbb{C}^{d\times d}$, for certain values of $d, m \in \mathbb{N}$.

By Proposition 3.2.5, we are interested in a way to prove that $Q \notin \overline{\Omega}_P$. By Theorem A.1.9.(1), the set Ω_P is locally closed and the following classical result implies that $\overline{\Omega}_P$ is also Zariski closed, hence an affine variety.

3.3.3 Theorem ([Kra85, AI.7.2]). Let X be a complex variety. For a constructible set $U \subseteq X$, the closure of U in the Euclidean and in the Zariski topology coincide. ■

It is the goal to show that $Q \notin \overline{\Omega}_P$. Assuming the converse, we get

$$Q \in \overline{P \circ \mathrm{GL}(W)} \iff Q \circ \mathrm{GL}(W) \subseteq \overline{P \circ \mathrm{GL}(W)} \iff \overline{Q \circ \mathrm{GL}(W)} \subseteq \overline{P \circ \mathrm{GL}(W)}.$$

This would imply that there is a surjection $\pi \colon \mathbb{C}[\overline{\Omega}_P] \twoheadrightarrow \mathbb{C}[\overline{\Omega}_Q]$ of the corresponding coordinate rings. Via the induced action of $\mathrm{GL}(W)$ on these rings (see Remark A.1.5), π is also a morphism of $\mathrm{GL}(W)$-modules. Choose a basis $W \cong \mathbb{C}^n$ and by Paragraph A.2.8, we can write

$$\mathbb{C}[\overline{\Omega}_P] = \bigoplus_{\lambda \in \Lambda_n^+} \mathbb{V}(\lambda)^{\underline{\mathrm{oc}}_P(\lambda)}.$$

We call $\underline{\mathrm{oc}}_P(\lambda)$ the **orbit closure coefficient** of P at λ. Since π is a GL_n-equivariant morphism, Schur's Lemma (Lemma A.2.2) implies that $\underline{\mathrm{oc}}_Q(\lambda) \leq \underline{\mathrm{oc}}_P(\lambda)$ for all $\lambda \in \Lambda_d^+$. Hence:

3.3.4 Proposition (The Flip). Let $P, Q \in \mathbb{C}[x_1, \ldots, x_n]_d$. If there exists a $\lambda \in \Lambda_n^+$ with $\underline{\mathrm{oc}}_Q(\lambda) > \underline{\mathrm{oc}}_P(\lambda)$, then $Q \notin \overline{\Omega}_P$. ■

This observation is called the "flip" because it allows us to prove the *non-existence* of efficient algorithms by proving the *existence* of certain objects.

3.3.5 Remark. It is quite unlikely that Proposition 3.3.4 is sufficient to separate Q from $\overline{\Omega}_P$ for general P and Q: The GL_n-module structure of the coordinate ring of an affine variety does not define the variety uniquely, see Example 3.3.6 below. It was kindly communicated to the author by Michel Brion and further related developments can be found in [Bri11]. ●

3.3.6 Example. Consider $G = \mathrm{SL}_2$ acting on the space $V = \mathbb{C}[x, y]_2$ of quadratic forms in two variables by precomposition. Let $\mathbb{C}[V] = \mathbb{C}[a, b, c]$ where we write a quadratic form as $a \cdot x^2 + b \cdot 2xy + c \cdot y^2$. The discriminant of quadratic forms gives a G-invariant regular function $\Delta := b^2 - ac \in \mathbb{C}[V]$. For any $t \in \mathbb{C}$, the polynomial $\Delta_t := \Delta - t$ is also G-invariant and the fibers $V_t := \{Q \in V \mid \Delta(Q) = t\} = \mathrm{Z}(\Delta_t) \subseteq V$ are closed

G-orbits in V, see [Kra85, II.3.3, Beispiel 1, p. 102]. The variety V_t is singular if and only if $t = 0$, so for example V_1 is not isomorphic to V_0. However, we claim that the G-module structure of the coordinate ring does not depend on t. Indeed, let $I \subseteq \mathbb{C}[V]$ be the ideal generated by Δ_t. Since Δ_t is G-invariant of degree 2, we have $\mathbb{C}[V]_{d-2} \cong I_d$ as G-modules for all $d \in \mathbb{Z}$, where the isomorphism is given by $P \mapsto P \cdot \Delta_t$. Hence, the G-module structure of I does not depend on the choice of t and in particular, the resulting module structure of $\mathbb{C}[V_t] = \mathbb{C}[V]/I$ is independent of t. ●

3.4 Orbit and Orbit Closure

For this chapter, let W be a $\mathbb{C}$-vector space of finite dimension with some choice of a basis $W \cong \mathbb{C}^n$. Usually, $n = d^2$ and W is a space of $d \times d$ matrices. We write $\mathrm{GL}(W)$ when coordinates are not required and GL_n otherwise.

We study the variety $\Omega_P = P \circ \mathrm{GL}(W)$ for some homogeneous form $P \in \mathbb{C}[W]_d$. This is the natural first step towards understanding its closure. By Theorem A.1.9, we are interested in the stabilizer group $G_P := \{g \in \mathrm{GL}(W) \mid P \circ g = P\}$ of P. The following two results justify that for the rest of this section, we assume G_P to be reductive and consequently, Ω_P is an affine variety with coordinate ring $\mathbb{C}[\mathrm{GL}(W)]^{G_P}$. For a matrix $A \in \mathbb{C}^{d \times d}$, we denote by A^{t} its **transpose**.

3.4.1 Theorem ([Fro97]). Let $W = \mathbb{C}^{d\times d}$ and let $\det_d \in \mathbb{C}[W]_d$ be the determinant. The stabilizer group $G_{\det_d}$ is reductive of dimension $2(d^2 - 1)$. Moreover,

(1) The identity component of the stabilizer of $\det_d$ is the group

$$G^{\circ}_{\det_d} = \{a \in \mathrm{GL}(W) \mid \exists S, T \in \mathrm{SL}_d \colon \forall B \in W \colon a(B) = SBT\}.$$

(2) Denoting by $t \in \mathrm{GL}(W)$ the map $t(B) := B^{\mathrm{t}}$, the group $G_{\det_d}$ consists of the two connected components $G^{\circ}_{\det_d}$ and $t \circ G^{\circ}_{\det_d}$. ■

3.4.2 Theorem ([Bot67; MM62]). Let $W = \mathbb{C}^{m\times m}$ with $m \geq 3$ and let $\mathrm{per}_m \in \mathbb{C}[W]_m$ be the permanent. The stabilizer group G_{per_m} is reductive of dimension $2(m-1)$. Moreover,

(1) Denoting by $\Delta_m \subseteq \mathrm{SL}_m$ the subgroup of diagonal matrices where the product of all entries on the diagonal equals 1,

$$G^{\circ}_{\mathrm{per}_m} = \{a \in \mathrm{GL}(W) \mid \exists S, T \in \Delta_m \colon \forall B \in W \colon a(B) = SBT\}.$$

(2) We embed $\mathfrak{S}_m \subseteq \mathrm{GL}_m$ as the subgroup of permutation matrices. For $\sigma, \tau \in \mathfrak{S}_m$, let $c^{\tau}_{\sigma} \in \mathrm{GL}(W)$ be the map $c^{\tau}_{\sigma}(B) = \sigma \circ B \circ \tau$ and let $G^{\sigma,\tau}_{\mathrm{per}_m} := c^{\tau}_{\sigma} \circ G^{\circ}_{\mathrm{per}_m}$.
Denoting by $t \in \mathrm{GL}(W)$ the map $t(B) := B^{\mathrm{t}}$, the group G_{per_m} consists of the $2m!^2$ connected components $t^k \circ G^{\sigma,\tau}_{\mathrm{per}_m}$ for $k \in \{0, 1\}$ and $\sigma, \tau \in \mathfrak{S}_m$. ■

$\mathbb{C}[\Omega_P]$ is also a G-module and the **orbit coefficients** $\mathrm{oc}_P(\lambda)$ are defined by

$$\mathbb{C}[\Omega_P] = \bigoplus_{\lambda \in \Lambda_n^+} \mathbb{V}(\lambda)^{\mathrm{oc}_P(\lambda)}.$$

Since Ω_P is an open, affine, GL_n-invariant subset of $\overline{\Omega}_P$, the restriction morphism $\mathbb{C}[\overline{\Omega}_P] \to \mathbb{C}[\Omega_P]$ is an injective morphism of GL_n-modules. By Schur's Lemma (Lemma A.2.2), we therefore have $\underline{\mathrm{oc}}_P(\lambda) \leq \mathrm{oc}_P(\lambda)$ for all $\lambda \in \Lambda_n^+$. While the orbit closure coefficients are quite elusive, the orbit coefficients admit a nice description when G_P is reductive:

3.4.3 Proposition. For $P \in \mathbb{C}[x_1, \ldots, x_n]$ with reductive stabilizer G_P and $\lambda \in \Lambda_n^+$, we have $\mathrm{oc}_P(\lambda) = \dim(\mathbb{V}(\lambda)^{G_P})$.

Proof. Recall from Theorems A.1.9 and A.2.7 that

$$\mathbb{C}[\Omega_P] = \mathbb{C}[\mathrm{GL}_n]^{G_P} = \bigoplus_{\lambda \in \Lambda_n^+} \mathbb{V}(\lambda)^* \otimes \mathbb{V}(\lambda)^{G_P}.$$

By Proposition A.2.10 and because $\lambda \mapsto \lambda^*$ defines an involution on Λ_n^+, we get

$$\mathbb{C}[\Omega_P] = \bigoplus_{\lambda \in \Lambda_n^+} \mathbb{V}(\lambda) \otimes (\mathbb{V}(\lambda)^*)^{G_P}.$$

The statement follows by applying Proposition A.2.12 to the representation $\mathbb{V}(\lambda)^*$ of the reductive group G_P. ∎

The orbit coefficients of the determinant and the padded permanent have been analyzed in [Bür+11].

3.4.1 Characterization by the Stabilizer

It is crucial to determine whether determinant and padded permanent suffer from the problem raised in Remark 3.3.5. The following special case of a result due to Larsen and Pink is interesting in this context. We implicitly use Proposition 3.4.3.

3.4.4 Theorem ([LP90]). Let $P \in \mathbb{C}[x_1, \ldots, x_n]_d$ be such that G_P is semisimple and connected. If $\varrho\colon G_P \hookrightarrow \mathrm{GL}_n$ is an irreducible G_P-representation, then G_P and ϱ are uniquely determined by the orbit coefficients $\mathrm{oc}_P\colon \Lambda_n^+ \to \mathbb{N}$, up to an isomorphism of groups and representations, respectively. ∎

The theorem is interesting because it implies that for certain polynomials orbit coefficients are sufficient to separate orbits. Mulmuley and Sohoni have shown that the determinant and the permanent are characterized by their stabilizer [MS01] in the following sense:

3.4.5 Definition. Let G be an algebraic group acting linearly on a $\mathbb{C}$-vector space W. A point $P \in W$ is said to be **characterized by its stabilizer** if $W^{G_P} = \mathbb{C} \cdot P$. ●

With this notion, we have:

3.4.6 Corollary. Assume that P and Q satisfy the assumptions of Theorem 3.4.4, we have $Q \notin \Omega_P$ and P is characterized by its stabilizer. Then, there exists a $\lambda \in \Lambda_n^+$ with $\mathrm{oc}_Q(\lambda) \neq \mathrm{oc}_P(\lambda)$.

Proof. We assume that P and Q have the same orbit coefficients and deduce $Q \in \Omega_P$. By Theorem 3.4.4, $G_P = g^{-1} \cdot G_Q \cdot g = G_{Q \circ g}$ for some $g \in \mathrm{GL}_n$, this is the definition of the two representations being isomorphic. Hence, $Q \circ g \in \mathbb{C}[x_1, \ldots, x_n]_d^{G_P} = \mathbb{C} \cdot P$ because P is characterized by its stabilizer. In other words, $Q \in \Omega_P$. ■

Unfortunately, the stabilizer groups of determinant and permanent are both disconnected, and only $G^\circ_{\det_d}$ is semisimple, $G^\circ_{\mathrm{per}_m} \cong \mathbb{G}_\mathrm{m}^{2(m-1)}$ is a torus, therefore only reductive. It has been established in [AYY13, Thm. 1.5] that the semisimplicity hypothesis in the general theorem by Larsen and Pink [LP90] cannot be dropped.

In the literature, the map $\lambda \mapsto \mathrm{oc}_P(\lambda)$ is referred to as the dimension datum of G_P. One can study the much more general case that G is an algebraic group and $H \subseteq G$ is a closed subgroup. The dimension datum of this inclusion is the map $\mathrm{Irr}(G) \to \mathbb{N}$ given by $\mathbb{V} \mapsto \dim(\mathbb{V}^H)$. Proposition 3.4.3 states that this is equivalent to the map $\mathrm{oc}_P \colon \Lambda_n^+ \to \mathbb{N}$ in our situation. The question of how much information about the inclusion $H \subseteq G$ is encoded in the dimension datum is an active and recent area of research, for some advances see [Yu16].

It seems that there is no version of Theorem 3.4.4 which is taylored to the situation we consider here, yet. We note however that the stabilizers of determinant and permanent act irreducibly on $W = \mathbb{C}^{d \times d}$.

3.4.7 Proposition. The space $W = \mathbb{C}^{d \times d}$ is an irreducible $G^\circ_{\det_d}$-representation and in particular, an irreducible $G_{\det_d}$-representation.

Proof. Since $W \cong \mathbb{C}^d \otimes \mathbb{C}^d$, it is an irreducible $(\mathrm{SL}_d \times \mathrm{SL}_d)$-representation [FH04, Exercise 2.36]. This is precisely the action of $G^\circ_{\det_d}$. ■

3.4.8 Proposition. The space $W = \mathbb{C}^{m \times m}$ is an irreducible G_{per_m}-representation.

Proof. Note first that $W = \sum_{i=1}^m \sum_{j=1}^m \mathbb{C} \cdot E_{ij}$ where E_{ij} is the matrix with entry 1 in position (i,j) and 0 everywhere else. Using notation from Theorem 3.4.2, the group $\Delta_m \times \Delta_m \cong \mathbb{G}_\mathrm{m}^{m-1} \times \mathbb{G}_\mathrm{m}^{m-1}$ is a torus and $\mathbb{C} \cdot E_{ij}$ is a weight space for the corresponding action by simultaneous left and right multiplication. This is the action of $G^\circ_{\mathrm{per}_m}$. Let now $U \subseteq W$ be some nonzero, G_{per_m}-stable subspace of W. Then, U is in particular stable under $G^\circ_{\mathrm{per}_m}$ and must therefore be a direct sum of certain weight spaces. Hence, there are indices i and j with $E_{ij} \in U$. By Theorem 3.4.2, arbitrary row and column permutations are in G_{per_m}, so $E_{ij} \in U$ for *all* $1 \le i, j \le m$. Hence, $W = U$. ■

Remark. While this sounds good, closer inspection reveals that the padded permanent fails to retain the desired property. This is related to the padding variable, as announced in Remark 3.3.1.

Let $m < d$ be natural numbers and $Q := \mathrm{pp}_{d,m}$ the padded permanent. Then, the space $W = \mathbb{C}^{d\times d}$ is certainly not an irreducible G_Q-representation: The one-dimensional space of matrices with zeros everywhere except in position (d, d) is invariant under G_Q due to the description of G_Q in [Bür+11, 5.6]. ○

Remark. Note that $W = \mathbb{C}^{m\times m}$ is *not* an irreducible $G^\circ_{\mathrm{per}_m}$-representation. ○

Even if we ignore all these problems for now, it remains an open question whether the orbit *closure* and its embedding are uniquely defined by the orbit closure coefficients. This is the important question of whether or not Proposition 3.3.4 is sufficient:

3.4.9 Question. Let $P, Q \in \mathbb{C}[x_1, \dots, x_n]_d$ have reductive stabilizer and $Q \notin \overline{\Omega}_P$. Assume that P is irreducible, characterized by its stabilizer, and $\mathbb{C}^n$ is irreducible as a G_P-module.

(1) Does there exist a $\lambda \in \Lambda_n^+$ with $\underline{\mathrm{oc}}_Q(\lambda) > \underline{\mathrm{oc}}_P(\lambda)$?

(2) The same question, but under the additional assumption that $Q = x^{d-m}\tilde{Q}$ is the product an irreducible polynomial $\tilde{Q} \in \mathbb{C}[x_1, \dots, x_n]_m$ of degree $m < d$ with a power of a linear form x, such that $\tilde{Q}$ is characterized by its stabilizer and $\mathbb{C}^n$ is irreducible as a $G_{\tilde{Q}}$-module.

(3) The same question, but even more specifically with $P = \det_d$ and $Q = \mathrm{pp}_{d,m}$.

Chapter 4

Occurrence Obstructions

The Flip (Proposition 3.3.4) offers a way to prove noncontainment of orbit closures. For the polynomials $P = \det_d$ and $Q = \mathrm{pp}_{d,m}$ it is proposed in [MS01; MS08] to exhibit weights λ which satisfy $\underline{\mathrm{oc}}_P(\lambda) = 0$ and $\underline{\mathrm{oc}}_Q(\lambda) \neq 0$. This implies $\underline{\mathrm{oc}}_Q(\lambda) > \underline{\mathrm{oc}}_P(\lambda)$ and therefore $Q \notin \overline{\Omega}_P$ by Proposition 3.3.4. Such weights λ are called **occurrence obstructions**.

We present in this chapter the results of the previously published [BIH17]. The paper features a result (Theorem 4.1.2) about the nature of occurrence obstructions which suggests that they are quite rare. In the subsequent works [IP16; BIP16], Bürgisser, Ikenmeyer, and Panova strengthened this result considerably, eventually proving that occurrence obstructions cannot be used to prove Conjecture 3.2.4. Their method relies only on the fact that we consider a padded polynomial, see also Remark 3.3.1. This is by far the biggest clue to date that the padding variable, while computationally harmless, causes severe problems for the geometry.

The auxiliary Propositions 4.3.5 and 4.3.8 remain of independent interest: They provide an unconditional version of a related statement by Kumar, even though the bounds are not particularly impressive.

In what follows, we will denote by $\mathscr{D}_d := \overline{\Omega}(\det_d)$ and $\mathscr{P}_{d,m} := \overline{\Omega}(\mathrm{pp}_{d,m})$ the orbit closures of determinant and padded permanent. Let also $n := d^2$. Both $\mathscr{D}_d$ and $\mathscr{P}_{d,m}$ are closed subvarieties of the N-dimensional affine space $\mathbb{A}^N = \mathbb{C}[x_{ij} \mid 1 \leq i,j \leq d]_d$ of degree d homogeneous forms on $\mathbb{C}^{d\times d}$, where we define $N := \binom{n+d-1}{d}$. The group $\mathrm{GL}_n \cong \mathrm{GL}(\mathbb{C}^{d\times d})$ acts on $\mathbb{A}^N$ by precomposition.

4.1 Weight Semigroups

We use notation from Section A.3 in this section. Let $Z \subseteq \mathbb{A}^N$ be any GL_n-invariant closed subvariety. The group GL_n acts on the coordinate ring $\mathbb{C}[Z]$ of Z via precomposition. We are interested in the set of irreducible GL_n-representations occurring in $\mathbb{C}[Z]$ and define

$$\Lambda^+(Z) := \{\lambda \in \Lambda_n^+ \mid \mathbb{V}(\lambda)^* \subseteq \mathbb{C}[Z]\}\,. \tag{1}$$

It is known that $\Lambda^+(Z)$ is a finitely generated submonoid of $\underline{\Lambda}_n^+$, cf. [Bri87]. We are mainly interested in the case where $Z = \overline{\Omega}_P$ is an orbit closure, in this case it is easy to see that $|\lambda| := \lambda_1 + \cdots + \lambda_n \in d\mathbb{Z}$ for any $\lambda \in \Lambda^+(Z)$. We furthermore denote by $\ell(\lambda) := \min\{k \mid \lambda_k = 0\}$ the **length** of λ. If P depends on ℓ variables only, then it is known that $\ell(\lambda) \leq \ell$ for all $\lambda \in \Lambda^+(Z)$, cf. [Bür+11, §6.3].

An occurrence obstruction is some $\lambda \in \Lambda^+(\mathscr{P}_{d,m}) \setminus \Lambda^+(\mathscr{D}_d)$. If we want to prove at least $\underline{\mathrm{dc}}(\mathrm{per}_m) > m^2 + 1$ by exhibting an occurrence obstruction, then we may assume that $d \geq m^2 + 1$. Since any λ in $\Lambda^+(\mathscr{P}_{d,m})$ satisfies $|\lambda| \in d\mathbb{Z}$ and $\ell(\lambda) \leq m^2 + 1 \leq d$, we have to look for such partitions λ outside of $\Lambda^+(\mathscr{D}_d)$.

Before stating our main result, we need to introduce the concept of saturation, whose relevance for geometric complexity was already pointed out in [Mul07], see also [BOR09].

For the following compare [MS05, §7.3]. Let S be a submonoid of a free abelian group F and $\langle S \rangle$ the group generated by S. We call S **saturated** if

$$\forall \lambda \in \langle S \rangle : \forall k \in \mathbb{N} \colon k\lambda \in S \Rightarrow \lambda \in S.$$

The **saturation** $\mathrm{Sat}(S)$ of S is defined as the smallest saturated submonoid of F containing S. It can also be characterized as the intersection of $\langle S \rangle$ with the rational cone generated by S. An element in $\mathrm{Sat}(S) \setminus S$ is called a **gap** of S. The reason for this naming becomes apparent from a simple example:

4.1.1 Example. Consider $S = \mathbb{N}^2 \setminus \{(0,1),(1,0)\}$, which has $\mathbb{N}^2$ as its saturation. Replacing S by $\mathrm{Sat}(S)$ means filling up the gaps $(0,1),(1,0)$. Generally, understanding monoids is difficult due to the presence of gaps. ●

We can now state the main theorem of this chapter:

4.1.2 Theorem. The saturation of $\Lambda^+(\mathscr{D}_d)$ contains the set

$$\left\{\lambda \in \underline{\Lambda}_n^+ \mid \ell(\lambda) \leq d,\ |\lambda| \in d\mathbb{Z}\right\},$$

provided that $d > 2$. Hence, occurrence obstructions must be gaps of $\Lambda^+(\mathscr{D}_d)$.

4.1.1 Related Work

Our work is closely related to a result by Shrawan Kumar [Kum15]. A **latin square** of size d is an $d \times d$ matrix with entries from $\{1, \ldots, d\}$ such that in each row and in each column each number occurs exactly once. The sign of a latin square is defined as the product of the signs of all the row and column permutations. Depending on the sign, we can speak about even and odd latin squares. The Alon-Tarsi conjecture [AT92] states that the number of even latin squares of size d is different from the number of odd latin squares of size d, provided d is even. The Alon-Tarsi conjecture is known to be true if $d = p \pm 1$ where p is a prime [Dri98; Gly10].

4.1.3 Theorem (Kumar). *Let d be even. If the Alon-Tarsi conjecture for $d \times d$ latin squares holds, then $d\lambda \in \Lambda^+(\mathscr{D}_d)$ for all partitions λ such that $\ell(\lambda) \le d$.*

The above two theorems complement each other. Theorem 4.1.2 is unconditional and also provides information about the group generated by $\Lambda^+(\mathscr{D}_d)$. Theorem 4.1.3 is conditional, but gives a very tight bound on the stretching factor, which is d if the Alon-Tarsi conjecture holds. The proofs of both theorems focus on the the d-th Chow variety (see Section 4.3 for details), but otherwise proceed differently. Our proof also gives information on the stretching factor in terms of certain degrees related to the normalization the Chow variety, but we were so far unable to bound it in a reasonable way, see Proposition 4.3.5.

4.2 Saturations of Weight Semigroups of Varieties

We consider the following situation. $G := \mathrm{GL}_n(\mathbb{C})$ is the complex general linear group and U denotes its subgroup consisting of the upper triangular matrices. V is a finite dimensional $\mathbb{C}$-vector space and a rational G-module such that scalar multiples of the unit matrix $\mathbb{I}_n \in G$ act on V by nontrivial homotheties, i.e., there is a nonzero $d \in \mathbb{Z}$ such that $P \circ t\mathbb{I}_n = t^d P$ for $t \in \mathbb{C}^\times$ and $P \in V$. Further, Z denotes a G-invariant, irreducible, locally closed nonempty subset of V. Then Z is closed under multiplication with scalars in $\mathbb{C}^\times$ by our assumption on the G-action.

We consider the induced action of G on the coordinate ring $\mathbb{C}[Z]$ of Z defined as in Remark A.1.5. We will denote this induced action from the left and by a dot, i.e., $(g.f)(P) = f(P \circ g^{-1})$ for $P \in Z$, $f \in \mathbb{C}[Z]$ and $g \in G$. As in (1) we define the monoid $\Lambda^+(Z)$ of representations of the G-variety Z. We shall interpret $\Lambda^+(G)$ as a subset of $\mathbb{Z}^n$ and denote by $\Lambda(Z) := \langle \Lambda^+(Z) \rangle$ the group generated by $\Lambda^+(Z)$. Moreover, we denote by $\mathrm{cone}_{\mathbb{Q}}(\Lambda^+(Z))$ the rational cone generated by $\Lambda^+(Z)$, that is,

$$\mathrm{cone}_{\mathbb{Q}}(\Lambda^+(Z)) := \left\{ k^{-1}\lambda \;\middle|\; k \in \mathbb{N}, \lambda \in \Lambda^+(Z) \right\} \subseteq \mathbb{Q}^n.$$

It is easy to check that the saturation of $\Lambda^+(Z)$ is obtained as

$$\mathrm{Sat}(\Lambda^+(Z)) = \Lambda(Z) \cap \mathrm{cone}_{\mathbb{Q}}(\Lambda^+(Z)). \tag{2}$$

We denote by $\mathrm{Frac}(R)$ the field of fractions of an integral ring R. We have an induced G-action on the field of fractions $\mathbb{C}(Z) := \mathrm{Frac}(\mathbb{C}[Z])$ and denote by $\mathbb{C}(Z)^U$ its subfield of U-invariants. Recall that a highest weight vector is a U-invariant weight vector. The following lemma is well known, but we include its proof for completeness.

4.2.1 Lemma. We have $\mathrm{Frac}(\mathbb{C}[Z]^U) = \mathbb{C}(Z)^U$. Moreover, for a highest weight vector $f \in \mathbb{C}(Z)^U$, there exist highest weight vectors $p, q \in \mathbb{C}[Z]^U$ such that $f = p/q$.

Proof. The inclusion $\mathrm{Frac}(\mathbb{C}[Z]^U) \subseteq \mathbb{C}(Z)^U$ is obvious. Now let $f \in \mathbb{C}(Z)^U$ and consider the ideal $J := \{q \in \mathbb{C}[Z] \mid qf \in \mathbb{C}[Z]\}$ of $\mathbb{C}[Z]$. Since $J \neq 0$ we have $J^U \neq 0$, cf. [Hum98, §17.5]. Choose a nonzero $q \in J^U$. Then $p := qf \in \mathbb{C}[Z]^U$ and $f = p/q$, hence $f \in \mathrm{Frac}(\mathbb{C}[Z]^U)$.

If $f \in \mathbb{C}(Z)^U$ is a weight vector, we can argue as before, choosing $q \in J^U$ as a highest weight vector. Then $p := qf$ is a highest weight vector in $\mathbb{C}[Z]$. The assertion follows. ■

4.2.2 Remark. If $Y \neq \emptyset$ is a G-invariant open subset of Z, then $\Lambda^+(Z) \subseteq \Lambda^+(Y)$ and $\Lambda(Z) = \Lambda(Y)$. This follows immediately from Lemma 4.2.1. ●

Suppose now that Z is a closed subset of V, hence an affine variety. Then we have an induced G-action on the normalization $\mathrm{N}(Z)$ and the canonical map $\pi\colon \mathrm{N}(Z) \to Z$ is G-invariant. Indeed, the integral closure R of $\mathbb{C}[Z]$ in $\mathbb{C}(Z)$ is G-invariant and π corresponds to the inclusion $\mathbb{C}[Z] \hookrightarrow R$. By construction, we can identify $\mathbb{C}(\mathrm{N}(Z))$ with $\mathbb{C}(Z)$. Note that $\Lambda^+(Z) \subseteq \Lambda^+(\mathrm{N}(Z))$ since π is surjective.

4.2.3 Proposition. We have $\Lambda(\mathrm{N}(Z)) = \Lambda(Z)$ and $\mathrm{Sat}(\Lambda^+(\mathrm{N}(Z))) = \mathrm{Sat}(\Lambda^+(Z))$. More precisely, assume that $\mathbb{C}[\mathrm{N}(Z)]$ is generated as a $\mathbb{C}[Z]$-module by r elements. Then for all $\lambda \in \Lambda^+(\mathrm{N}(Z))$, there is some $k < r$ such that $(r-k) \cdot \lambda \in \Lambda^+(Z)$.

Proof. Let $\lambda \in \Lambda^+(\mathrm{N}(Z))$ and $f \in \mathbb{C}[\mathrm{N}(Z)]^U$ be a highest weight vector of weight λ. Then $f \in \mathbb{C}(Z)^U$ and Lemma 4.2.1 shows the existence of highest weight vectors $p, q \in \mathbb{C}[Z]^U$, say with the weights $\mu, \nu \in \Lambda^+(Z)$, respectively, such that $f = p/q$. Therefore $\lambda = \mu - \nu \in \Lambda(Z)$. This shows the equality for the groups.

Due to (2), it suffices to prove that $\mathrm{cone}_{\mathbb{Q}}(\Lambda^+(Z)) = \mathrm{cone}_{\mathbb{Q}}(\Lambda^+(\mathrm{N}(Z)))$. Suppose $f \in \mathbb{C}[\mathrm{N}(Z)]$ is a highest weight vector of weight λ. Since f is integral over $\mathbb{C}[Z]$, there are $e \in \mathbb{N}$ and $a_0, \dots, a_{e-1} \in \mathbb{C}[Z]$ such that such that

$$f^e + \sum_{i=0}^{e-1} a_i f^i = 0. \tag{3}$$

We assume that the degree e is the smallest possible.

Note that e is at most the size of a generating set of $\mathbb{C}[\mathrm{N}(Z)]$ as an $\mathbb{C}[Z]$-module, as follows from the classical theory of integral extensions, see [AM69, Prop. 5.1 and the proof of Prop. 2.4].

Consider the weight decomposition $a_i = \sum_\mu a_{i,\mu}$ of a_i, where $a_{i,\mu}$ has the weight μ. Then $a_{i,\mu} f^i$ has the weight $\mu + i\lambda$. Moreover, f^e has the weight $e\lambda$. Since the component of weight $e\lambda$ in (3) must vanish, we have

$$f^e + \sum_{i=0}^{e-1} a_{i,(e-i)\lambda} \cdot f^i = 0.$$

As the degree e is the smallest possible, the above is the minimal polynomial of f. Applying any element $u \in U$ to the above equation and using $u.f = f$, we get the identity $f^e + \sum_{i=0}^{e-1}(u.a_{i,(e-i)\lambda})f^i = 0$. The uniqueness of the minimal polynomial implies that $u.a_{i,(e-i)\lambda} = a_{i,(e-i)\lambda}$ for all i. Hence $a_{i,(e-i)\lambda}$ is a highest weight vector, provided it is nonzero. Since there exists $i < e$ with $a_{i,(e-i)\lambda} \neq 0$, we see that $(e-i)\lambda \in \Lambda^+(\mathbb{C}[Z])$ for this particular i. We conclude that $\lambda \in \operatorname{cone}_{\mathbb{Q}}(\mathbb{C}[Z])$. ∎

4.2.4 Example. If we consider instead the torus $G = (\mathbb{C}^\times)^d$ we can identify Λ_G^+ with $\mathbb{Z}^d$. Let $S \subseteq \mathbb{Z}^d$ be a finitely generated submonoid and consider the finitely generated subalgebra $\mathbb{C}[S] := \bigoplus_{s\in S}(\mathbb{C} \cdot x_1^{s_1} \cdots x_d^{s_d})$ of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$. If we interpret $\mathbb{C}[S]$ as the coordinate ring of an affine variety Z, then we have a G-action on Z and $\Lambda^+(Z)$ can be identified with S (Z is called a toric variety). It is known that $\mathbb{C}[\operatorname{Sat}(S)]$ equals the integral closure of $\mathbb{C}[S]$ in $\operatorname{Frac}(\mathbb{C}[S])$, cf. [MS05, Prop. 7.25, p. 140]. Thus the affine variety corresponding to $\operatorname{Sat}(S)$ equals the normalization of Z. This illustrates Proposition 4.2.3 in the special case of toric varieties. ●

4.3 Proof of Main Results

Write $W := \mathbb{C}^n$, $G := \operatorname{GL}(W)$ and consider the symmetric power $V := \operatorname{Sym}^n W^*$ with is natural G-action. Using the chosen coordinates on W, we obtain an isomorphism $W \to W^*$ that we denote by $a \mapsto a^*$. We can interpret V as the space $\mathbb{C}[W]_n$ of degree n forms. Note that $\mathbb{C}[V]_k \cong \operatorname{Sym}^k \operatorname{Sym}^n W$ as G-modules. Let $x_1, \ldots, x_n \in W^*$ be a basis and consider the **universal monomial**

$$\mathrm{mn}_n := x_1 \cdots x_n \in V.$$

Clearly, Ω_{mn_n} consists of all symmetric products $a_1 \cdots a_n$ of n linearly independent linear forms a_i. We define the n-th **Chow variety**

$$\mathscr{C}_n := \mathrm{mn}_n \circ \operatorname{End}(W) = \{a_1 \cdots a_n \mid a_1, \ldots, a_n \in W^*\} \subseteq V.$$

The name comes from the fact that $\mathscr{C}_n$ is a special case of a *Chow variety*, see [GKZ94].

4.3.1 Lemma. We have $\mathscr{C}_n = \overline{\Omega}_{\mathrm{mn}_n}$.

Proof. The inclusion $\mathscr{C}_n \subseteq \overline{\Omega}_{\mathrm{mn}_n}$ follows since $\operatorname{GL}(W)$ is dense in $\operatorname{End}(W)$. For the converse inclusion, let $Q \in \overline{\Omega}_{\mathrm{mn}_n}$ be nonzero. By Theorem 3.3.3, it is the limit of a sequence $(t_k \cdot a_{1k} \cdots a_{nk})_{k\in\mathbb{N}}$, where $t_k \in \mathbb{C}^\times$ and $a_{ik} \in W^*$ are linearly independent with $\|a_{ik}\| = 1$. By compactness of the unit sphere in W^* we may assume that after passing to a subsequence, $(a_{ik})_{k\in\mathbb{N}}$ is convergent for all $1 \leq i \leq n$. Let $b_i := \lim_{k\to\infty} a_{ik}$. Then, $(\prod_{i=1}^n a_{ik})_{k\in\mathbb{N}}$ converges to $b_1 \cdots b_n \neq 0$. It follows easily that $(t_k)_{k\in\mathbb{N}}$ converges to some $t \in \mathbb{C}^\times$ and consequently, $Q = t \cdot b_1 \cdots b_n \in \mathscr{C}_n$. ∎

When we identify x_i with the variable x_{ii}, the Chow variety $\mathscr{C}_n$ is contained in $\mathscr{D}_n$ by mapping x_{ij} to 0 for $i \neq j$, c.f. [Lan15]. The basic strategy, as in Kumar [Kum15], is to replace $\mathscr{D}_n$ by the considerably simpler $\mathscr{C}_n$ and to exhibit elements in the monoid of representations of the latter. More specifically, we have $\Lambda^+(\mathscr{C}_n) \subseteq \Lambda^+(\mathscr{D}_n)$ and hence $\mathrm{Sat}(\Lambda^+(\mathscr{C}_n)) \subseteq \mathrm{Sat}(\Lambda^+(\mathscr{D}_n))$. Our main Theorem 4.1.2 is an immediate consequence of the following result.

4.3.2 Theorem. We have $\mathrm{Sat}(\Lambda^+(\mathscr{C}_n)) = \{\lambda \in \underline{\Lambda}_n^+ : |\lambda| \in n\mathbb{Z}\}$, provided $n > 2$.

According to Proposition 4.2.3, for proving this we may replace $\mathscr{C}_n$ by its normalization. It is crucial that the latter has an explicit description.

For the following arguments compare [Bri93] and [Lan15]. We will revisit them in Section 6.2. The symmetric group $\mathfrak{S}_n$ operates on the group

$$T_n := \{(t_1, \ldots, t_n) \in \mathbb{C}^n \mid t_1 \cdots t_n = 1\}$$

by permutation. The corresponding semidirect product $H_n := T_n \rtimes \mathfrak{S}_n$ acts on the space $W^n = W \times \ldots \times W$ by scaling and permutation. Note that this action commutes with the G-action. Consider the product map

$$\begin{aligned} \omega\colon W^n = W \times \ldots \times W &\longrightarrow \mathscr{C}_n \\ (a_1, \ldots, a_n) &\longmapsto a_1^* \cdots a_n^*, \end{aligned}$$

which is surjective and G-equivariant. Clearly, ω is invariant on H-orbits. Moreover, the fiber of a nonzero $Q \in \mathscr{C}_n$ is an H_n-orbit. This easily follows from the uniqueness of polynomial factorization.

The group G contains the subgroup $\{t \cdot \mathrm{id}_W \mid t \in \mathbb{C}^\times\} \cong \mathbb{C}^\times$ and this $\mathbb{C}^\times$-action induces a natural grading on the coordinate rings $\mathbb{C}[W^n]$ and $\mathbb{C}[\mathscr{C}_n]$. Since ω is G-equivariant, the corresponding comorphism $\omega^\sharp\colon \mathbb{C}[\mathscr{C}_n] \to \mathbb{C}[W^n]$ is in particular a homomorphism of graded $\mathbb{C}$-algebras. However, the $\mathbb{C}^\times$-action on $\mathbb{C}[\mathscr{C}_n]$ is not the canonical one because $t \in \mathbb{C}^\times$ acts by multiplication with the scalar t^n. A homogeneous element of degree kn in $\mathbb{C}[\mathscr{C}_n]$ is the restriction of a k-form on V.

The categorical quotient $W^n /\!\!/ H_n$ is defined as the affine variety that has as its coordinate ring the ring of H_n-invariants $\mathbb{C}[W^n]^{H_n}$, which is finitely generated and graded since H_n is reductive, cf. [Kra85]. The inclusion $\mathbb{C}[W^n]^{H_n} \hookrightarrow \mathbb{C}[W^n]$ defines a G-equivariant, surjective morphism $\pi\colon W^n \to W^n /\!\!/ H_n$. Since W^n is normal, the quotient $W^n /\!\!/ H_n$ is normal as well, see [Dol03, p. 45] for the easy proof. The map ω factors through a G-equivariant morphism

$$\phi\colon W^n /\!\!/ H_n \to \mathscr{C}_n, \tag{4}$$

due to the universal property of categorical quotients. Moreover, by construction, the fibers of ϕ over a nonzero $Q \in \mathscr{C}_n$ consist of just one element. The action of H_n

on $\mathbb{C}[W^n]$ is linear, therefore it respects the grading. It follows that the comorphism $\phi^\sharp\colon \mathbb{C}[\mathscr{C}_n] \to \mathbb{C}[W^n]^{H_n}$ is again a homomorphism of graded $\mathbb{C}$-algebras.

The following is shown in [Bri93, Prop., p. 351] and we give a proof in slightly different language in Section 6.3.

4.3.3 Lemma. The morphism $\phi\colon W^n /\!\!/ H_n \to \mathscr{C}_n$ is the normalization of $\mathscr{C}_n$. ∎

Furthermore,

4.3.4 Lemma. We have

$$\Lambda^+(W^n /\!\!/ H_n) = \left\{ \lambda \in \underline{\Lambda}_n^+ \;\middle|\; \exists k\colon |\lambda| = kn \text{ and } \mathbb{V}_G(\lambda) \subseteq \operatorname{Sym}^n \operatorname{Sym}^k \mathbb{C}^n \right\}.$$

Proof. We shall decompose the coordinate ring $\mathbb{C}[W^n /\!\!/ H_n]$ with respect to the G-action. We have $\mathbb{C}[W] = \bigoplus_{k\in\mathbb{N}} \operatorname{Sym}^k W^*$ and therefore

$$\mathbb{C}[W^n] = \mathbb{C}[W]^{\otimes n} = \bigoplus_{(k_1,\ldots,k_n)\in\mathbb{N}^n} \operatorname{Sym}^{k_1} W^* \cdots \otimes \operatorname{Sym}^{k_n} W^*.$$

Taking T_n-invariants yields

$$\mathbb{C}[W^n]^{T_n} = \bigoplus_{k\in\mathbb{N}} \operatorname{Sym}^k W^* \otimes \ldots \otimes \operatorname{Sym}^k W^*.$$

Taking $\mathfrak{S}_n$-invariants gives

$$\mathbb{C}[W^n /\!\!/ H_n] \cong \mathbb{C}[W^n]^{T_n \rtimes S_n} = \bigoplus_{k\in\mathbb{N}} \operatorname{Sym}^n \operatorname{Sym}^k W^*, \tag{5}$$

and the assertion follows. ∎

Recalling Theorem 4.1.3, we expect the stretching factor to be n. Without relying on the Alon-Tarsi conjecture however, the following exponential bound is the best we can currently provide.

4.3.5 Proposition. $\Lambda^+(\mathscr{C}_n)$ generates the rational cone $\{q \in \mathbb{Q}^n \mid q_1 \geq \cdots \geq q_n \geq 0\}$.

More precisely: Assume $n > 2$. For each partition λ with $\ell(\lambda) \leq n$ and $|\lambda| \in n\mathbb{Z}$, there is some number $k < n^{n^2-2n}$ such that we have $2k \cdot \lambda \in \Lambda^+(\mathscr{C}_n)$.

Proof. For the first statement, it is sufficient to show that $\operatorname{cone}_{\mathbb{Q}}(\Lambda^+(\mathscr{C}_n))$ contains any partition λ with $|\lambda| = nk$ and $\ell(\lambda) \leq n$. According to Proposition 4.2.3, the semigroups $\Lambda^+(\mathscr{C}_n)$ and $\Lambda^+(W^n /\!\!/ H_n)$ generate the same rational cone. So we need to show that λ lies in $\operatorname{cone}_{\mathbb{Q}}(\Lambda^+(W^n /\!\!/ H_n))$.

In [BCI11] (see [Ike12] for a simpler proof) it was shown that $\mathbb{V}_G(2\lambda)$ occurs in $\operatorname{Sym}^n \operatorname{Sym}^{2k}(\mathbb{C}^n)$. Thus Lemma 4.3.4 with Proposition 4.2.3 imply that λ lies in the rational cone generated by $\Lambda^+(W^n /\!\!/ H_n)$.

We will now make the above reference to Proposition 4.2.3 precise. Recall that the comorphism of ϕ from (4) is an integral extension $\phi^\sharp\colon \mathbb{C}[\mathscr{C}_n] \hookrightarrow \mathbb{C}[W^n]^{H_n}$ of graded $\mathbb{C}$-algebras. Note that the grading induced by G on these two rings is such that only degrees which are multiples of n contain nonzero elements. We therefore change the grading such that degree $n \cdot k$ becomes degree k. This means that the direct sum (5) is the grading of $\mathbb{C}[W^n]^{H_n}$ and the grading of $\mathbb{C}[\mathscr{C}_n]$ is the grading induced by the canonical grading induced by the polynomial ring $\mathbb{C}[V]$. Therefore, $\mathbb{C}[\mathscr{C}_n]$ is generated by elements of degree one.

By [DK02, Lemma 2.4.7], there is a system of parameters $y_0, \ldots, y_r \in \mathbb{C}[\mathscr{C}_n]_1$ from among sufficiently generic linear forms. This means that $R := \mathbb{C}[y_0, \ldots, y_r]$ is a polynomial ring in the y_i and $\mathbb{C}[\mathscr{C}_n]$ is a finite R-module. It follows that $\mathbb{C}[W^n]^{H_n}$ is integral over R and $y_0, \ldots, y_r$ is also a homogeneous system of parameters for $\mathbb{C}[W^n]^{H_n}$.

We note that $r+1 = \dim(W^n /\!\!/ H_n) = \dim(W^n) - \dim(H_n) = n^2 - n + 1$, so $r = n^2 - n$. Furthermore, $\mathbb{C}[W^n]^{H_n}$ is Cohen-Macaulay [DK02, Thm. 2.5.5] and by [DK02, Prop. 2.5.3], it follows that $\mathbb{C}[W^n]^{H_n}$ is free as an R-module, i.e., $\mathbb{C}[W^n]^{H_n} \cong R^D$ for some $D \in \mathbb{N}$. Since D is the number of generators of $\mathbb{C}[W^n]^{H_n}$ as an R-module, D is a (possibly rough) upper bound for the number of generators of $\mathbb{C}[W^n]^{H_n}$ as an $\mathbb{C}[\mathscr{C}_n]$-module. The second assertion follows from Proposition 4.2.3 as soon as we have verified that $D < n^{n^2-2n}$.

The Hilbert polynomial of R^D is a polynomial of degree r whose leading coefficient is equal to $\frac{D}{r!}$. Since (5) gives the grading of $\mathbb{C}[W^n]^{H_n}$, the Hilbert polynomial of $\mathbb{C}[W^n]^{H_n}$ is given, for sufficiently large k, by the map

$$k \mapsto \binom{\binom{k+n-1}{n-1} + n - 1}{n},$$

whose leading coefficient is $\frac{1}{n!(n-1)!^n}$. Since $r = n^2 - n$, we have $D = \frac{(n^2-n)!}{n!(n-1)!^n}$. We apply Stirling's approximation:

$$\forall n > 0: \quad 1 \leq \frac{n!}{\sqrt{2\pi} \cdot e^{-n} \cdot n^{n+\frac{1}{2}}} \leq e^{\frac{1}{12n}}$$

to the fraction D and obtain

$$\begin{aligned} D = \frac{(n^2-n)!}{n!(n-1)!^n} &\leq \frac{\sqrt{2\pi} \cdot e^{\frac{1}{12n}} \cdot (n^2-n)^{n^2-n+\frac{1}{2}} \cdot e^{n-n^2}}{\sqrt{2\pi} \cdot n^{n+\frac{1}{2}} \cdot e^{-n} \cdot \sqrt{2\pi}^n \cdot (n-1)^{(n-\frac{1}{2})n} \cdot e^{-(n-1)n}} \\ &= \left(\frac{e}{\sqrt{2\pi}}\right)^n \cdot e^{\frac{1}{12n}} \cdot \frac{n^{n^2-n+\frac{1}{2}} \cdot (n-1)^{n^2-n+\frac{1}{2}}}{n^{n+\frac{1}{2}} \cdot (n-1)^{n^2-\frac{n}{2}}} \\ &= \underbrace{\left(\left(\frac{e}{\sqrt{2\pi}}\right)^n \cdot e^{\frac{1}{12n}} \cdot (n-1)^{\frac{1-n}{2}}\right)}_{R(n)} \cdot n^{n^2-2n}. \end{aligned}$$

It is easy to see that $R(n)$ is monotonically decreasing for $n \geq 2$ and takes a value smaller than 1 for $n = 3$. Hence, for $n > 2$ we have $D < n^{n^2-2n}$. ■

We shall now determine $\Lambda(\mathscr{C}_n)$. Since $W^n /\!\!/ H_n$ is the normalization of $\mathscr{C}_n$, Proposition 4.2.3 tells us that $\Lambda(\mathscr{C}_n) = \langle \Lambda^+(W^n /\!\!/ H_n) \rangle$. The latter is described in terms of plethysms in Lemma 4.3.4.

Recall $W = \mathbb{C}^n$ and $G = \mathrm{GL}(W)$. We say that λ occurs in $\mathrm{Sym}^n \mathrm{Sym}^k W$ if $\mathbb{V}_G(\lambda)$ occurs as a submodule in the latter. We will also make use of the convenient notation $k \times d := (d, \ldots, d, 0, \ldots, 0) \in \mathbb{Z}^n$ for a rectangular partition with k rows of length d.

4.3.6 Lemma. Let $\ell \in \mathbb{N}$. If λ occurs in $\mathrm{Sym}^n \mathrm{Sym}^k W$, then $(1 \times \ell k) + \lambda$ occurs in $\mathrm{Sym}^{\ell+n} \mathrm{Sym}^k W$.

Proof. Let $W^U = \mathbb{C}w$, i.e., w is a highest weight vector of W with weight 1×1. Then $w^{\otimes \ell k} \in \mathrm{Sym}^\ell \mathrm{Sym}^k W$ is a highest weight vector of weight $(1 \times \ell k)$.

Let $f \in \mathrm{Sym}^n \mathrm{Sym}^k W$ be a highest weight vector of weight λ. Then the product $w^{\otimes \ell k} f \in \mathrm{Sym}^{\ell+n} \mathrm{Sym}^k W$ is a highest weight vector of weight $(1 \times \ell k) + \lambda$. ■

4.3.7 Lemma. Let $n \geq k \geq 2$, $d := k(k-1)/2$, and the partition μ of size k^2 be obtained by appending to $2 \times d$ a column of length k. Further, let λ denote the partition of size nk obtained by appending to μ a row of length $(n-k)k$. Then the partition λ occurs in $\mathrm{Sym}^n \mathrm{Sym}^k W$.

Proof. The GL_2-module $\Lambda^k \mathrm{Sym}^{k-1} \mathbb{C}^2$ is one-dimensional, since $\mathrm{Sym}^{k-1} \mathbb{C}^2$ is of dimension k. Hence it contains a nonzero SL_2-invariant. In other words, $2 \times d$ occurs in $\Lambda^k \mathrm{Sym}^{k-1} \mathbb{C}^2$. The "inheritance principle" states that $2 \times d$ occurs in $\Lambda^k \mathrm{Sym}^{k-1} \mathbb{C}^n$ (compare for instance [Ike12, Lemma 4.3.2]).

Cor. 6.4 in [MM15] implies that μ occurs in $\mathrm{Sym}^k \mathrm{Sym}^k W$. Finally, Lemma 4.3.6 implies the assertion. ■

4.3.8 Proposition. $\Lambda^+(\mathscr{C}_n)$ generates the group $\{\lambda \in \mathbb{Z}^n \;:\; |\lambda| \in n\mathbb{Z}\}$ if $n > 2$.

Proof. Using the software [BKT12], we checked that the partition $(2, 2, 0, \ldots, 0)$ occurs in $\mathrm{Sym}^2 \mathrm{Sym}^2 W$ and $(6, 3, 0, \ldots, 0)$ occurs in $\mathrm{Sym}^3 \mathrm{Sym}^3 W$. Using Lemma 4.3.6, we can conclude that $(2n-2, 2, 0, \ldots, 0)$ occurs in $\mathrm{Sym}^n \mathrm{Sym}^2 W$, and $(3n-3, 3, 0, \ldots, 0)$ occurs in $\mathrm{Sym}^n \mathrm{Sym}^3 W$ if $n \geq 3$. (We note that $(3, 3, 0, \ldots, 0)$ does not occur in $\mathrm{Sym}^2 \mathrm{Sym}^3 W$; this is the reason for the assumption $n > 2$.) From Lemma 4.3.4 we conclude that

$$\lambda^{(2)} := (n-1, 1, 0, \ldots, 0) = (3n-3, 3, 0, \ldots, 0) - (2n-2, 2, 0, \ldots, 0)$$

lies in the group Λ generated by $\Lambda^+(W^n /\!\!/ H_n)$. Clearly, $\lambda^{(1)} := (n, 0, \ldots, 0) \in \Lambda$.

For $3 \leq k \leq n$ let $\lambda^{(k)} \in \Lambda$ denote the partition from Lemma 4.3.7. Then we have $\ell(\lambda^{(k)}) = k$ and $\lambda_k^{(k)} = 1$. This easily implies that $\lambda^{(1)}, \ldots, \lambda^{(n)}$ generate the group $\tilde{\Lambda} := \{\lambda \in \mathbb{Z}^n : |\lambda| \in n\mathbb{Z}\}$. Since $\Lambda \subseteq \tilde{\Lambda}$ is obvious, we conclude that $\Lambda = \tilde{\Lambda}$. ■

Proof of Theorem 4.3.2. Use (2) with Propositions 4.3.5 and 4.3.8. ■

4.3.9 Remark. The assumption $n > 2$ in Theorem 4.3.2 is necessary. Indeed, we have $\mathscr{C}_2 = \operatorname{Sym}^2 \mathbb{C}^2$, and one can show that $\Lambda^+(\operatorname{Sym}^2 \mathbb{C}^2)$ generates the group $(2\mathbb{Z})^2$, compare [FH04, §11.2]. ●

Part II

Orbit Closures of Homogeneous Forms

Chapter 5

Preliminaries

Throughout Part II, we will consider the following situation. We work over the field $\mathbb{C}$ of complex numbers and $W \cong \mathbb{C}^n$ is a $\mathbb{C}$-vector space of dimension n, for which we sometimes assume some choice of coordinates $\mathbb{C}[W] = \mathbb{C}[x_1, \ldots, x_n]$, to identify $\mathrm{GL}(W)$ with $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$. The group $\mathrm{GL}(W)$ acts on the space $\mathbb{C}[W]_d = \mathrm{Sym}^d W^*$ of homogeneous forms of degree $d > 0$ on W by precomposition from the right:

$$\begin{aligned} \mathbb{C}[W]_d \times \mathrm{GL}(W) &\longrightarrow \mathbb{C}[W]_d \\ (P, g) &\longmapsto P \circ g \end{aligned}$$

We view $V := \mathbb{C}[W]_d \cong \mathbb{A}^N$ as an affine space of dimension $N = \binom{n+d+1}{d}$. Note that $\mathbb{C}[V]$ is simply a polynomial ring in the coefficients of all homogeneous degree d polynomials in n variables. We will study subvarieties of this space in the language of algebraic geometry, therefore we usually assume the Zariski topology on V.

This scenario is motivated by the GCT approach that we outlined in Chapter 3. In that context, one is interested only in special cases, primarily $W = \mathbb{C}^{d \times d}$ and the point $P = \det_d$. We put this into a slightly more general context here, one reason being that we have very little understanding of the determinant orbit closure, even for small values of d. For what little we know, see Chapter 8. To gain a better understanding of the phenomena that occur, we believe that easier examples should be studied first.

Another reason is also that the generality we present here offers an interesting and little explored mathematical problem: Classical geometric invariant theory offers the tools to study the entire orbit structure of V, but a classification seems out of reach for $n > 3$ and $d > 4$. We on the other hand pick specific forms $P \in V$ and study only the orbit structure of $\overline{\Omega}_P$, or even just the orbits that are of maximal dimension in a component of $\partial\Omega_P$. In this chapter, we make some general observations and elaborate on methods that we will apply in the chapters to come.

Unless explicitly stated otherwise, we only consider points $P \in V$ whose stabilizer $G_P = \{g \in \mathrm{GL}(W) \mid P \circ g = P\}$ is reductive, mostly because it simplifies the theory considerably and the condition is satisfied in our cases of primary interest, recall Theorems 3.4.1 and 3.4.2.

We write $\Omega_P := P \circ \mathrm{GL}(W) \subseteq V$ for the orbit of P and denote by $\overline{\Omega}_P \subseteq V$ its closure as in Definition 3.3.2. Recall also that by Theorem 3.3.3, $\overline{\Omega}_P$ is both the Euclidean and the Zariski closure of Ω_P. An element of $\partial\Omega_P$ is called a **degeneration** of P.

5.1 Conciseness

Let us call a form $P \in V = \mathbb{C}[W]_d$ **concise** if it is not stabilized by any noninvertible endomorphism, i.e.

$$\forall a \in \mathrm{End}(W)\colon (P \circ a = P) \Rightarrow (a \in \mathrm{GL}(W)).$$

We will give other characterizations of this property here, for example it means that there is no choice of coordinates $\mathbb{C}[W] = \mathbb{C}[x_1, \ldots, x_n]$ such that $P \in \mathbb{C}[x_1, \ldots, x_{n-1}]$. Informally, this may be expressed as "P uses all variables". We will also see that being concise is an open condition in Proposition 5.1.2.

5.1.1 Definition. Let $x \in W^* = \mathbb{C}[W]_1$ and $w \in W$. We define the **partial derivative** of $x^d \in \mathbb{C}[W]_d$ in direction w to be the form $\partial_w x^d := d \cdot x(w) \cdot x^{d-1}$, c.f. [Lan12, eq. (2.6.6)]. By linear extension, this defines $\partial_w P$ for any $P \in \mathbb{C}[W]_d$ because $\mathbb{C}[W]_d = \mathrm{Sym}^d W^*$ is spanned by powers of linear forms as a vector space [Mar73, Theorem 1.7]. If we have chosen coordinates $\mathbb{C}[W] = \mathbb{C}[x_1, \ldots, x_n]$, we set $\partial_i P := \partial_{e_i} P$ for $1 \leq i \leq n$, where $e_i \in W$ is the dual basis vector of $x_i \in W^*$. ●

5.1.2 Proposition. Let W be a vector space of finite dimension. A form $P \in \mathbb{C}[W]_d$ is concise if and only if the linear map $W \to \mathbb{C}[W]$, $w \mapsto \partial_w P$ is injective.

Proof. The case $d = 0$ is trivial, a constant form P is stabilized by any endomorphism, hence it is concise if and only if $\mathrm{GL}(W) = \mathrm{End}(W)$, which is equivalent to $W = \{0\}$. The case $d \geq 1$ follows from Lemma 5.1.6 below for the identity map $a = \mathrm{id}_W$. ■

5.1.3 Remark. In particular, *not* being concise is a polynomial condition on the coefficients of $P \in \mathbb{C}[W]_d$: It is given by the vanishing of all maximal minors of the linear map $W \to \mathbb{C}[W]_{d-1}$, $w \mapsto \partial_w P$. Hence, the set of all concise forms is a Zariski open subset of $\mathbb{C}[W]_d$. ●

5.1.4 Corollary. Let $P \in \mathbb{C}[x_1, \ldots, x_n]_d$. Then, P is concise if and only if the partial derivatives $\partial_1 P, \ldots, \partial_n P$ are linearly independent. ■

5.1.5 Remark. After choosing coordinates, a polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$ satisfies $x_i \in \operatorname{supp}(P)$ if and only if $\partial_i P \neq 0$. Hence, P is not concise if and only if there is a choice of coordinates such that one of the variables does not appear in the support of P at all. ●

For $a \in \operatorname{End}(W)$, we denote by $\operatorname{rk}(a) := \dim(a(W))$ its **rank**.

5.1.6 Lemma. Let $P \in \mathbb{C}[W]_d$ with $d \geq 1$. For $a \in \operatorname{End}(W)$, we consider the linear map $\delta_a \colon W \to \mathbb{C}[W]$, $w \mapsto \partial_w(P \circ a)$. The following conditions are equivalent:

(1) $\operatorname{rk}(a) \leq \operatorname{rk}(\delta_a)$.

(2) For any $b \in \operatorname{End}(W)$ with $P \circ a = P \circ b$, we have $\operatorname{rk}(a) \leq \operatorname{rk}(b)$.

Proof. We choose coordinates $\mathbb{C}[W] = \mathbb{C}[x_1, \dots, x_n]$ and set $Q := P \circ a$. Denote by

$$\nabla(Q) := (\partial_1 Q, \dots, \partial_n Q) \in \mathbb{C}[x_1, \dots, x_n]_{d-1}^{1 \times n}$$

the (row) vector of partial derivatives of Q. Let $v_1, \dots, v_k \in \mathbb{C}^n$ be a basis of the vector space $\{v \in \mathbb{C}^n \mid \nabla(Q) \cdot v = 0\}$, so $r := n - k$ is the dimension of the $\mathbb{C}$-vector space spanned by the $\partial_i Q$, i.e., $r = \operatorname{rk}(\delta_a)$.

Assume that (2) holds. We have to show that $\operatorname{rk}(a) \leq r$. Let $g \in \operatorname{GL}(W)$ be an invertible linear transformation that maps the i-th standard basis vector $e_i \in \mathbb{C}^n$ to v_i for $1 \leq i \leq k$. Denoting by a dot the product of matrices, the chain rule yields

$$\partial_i(Q \circ g) = \nabla(Q \circ g) \cdot e_i = (\nabla(Q) \cdot v_i) \circ g = 0 \quad \text{for} \quad 1 \leq i \leq k.$$

This implies that $Q \circ g \in \mathbb{C}[x_{k+1}, \dots, x_n]$. We consider the linear map $c \in \operatorname{End}(W)$ which maps $x_i \circ c = 0$ for $1 \leq i \leq k$ and $x_j \circ c = x_j$ for $j > k$. Then, $Q \circ g \circ c = Q \circ g$, therefore $P \circ agcg^{-1} = Q = P \circ a$. It follows that $r = \operatorname{rk}(c) \geq \operatorname{rk}(agcg^{-1}) \geq \operatorname{rk}(a)$ by our assumption.

Conversely, assume that (1) holds, so we have $\operatorname{rk}(a) \leq r$. Let $b \in \operatorname{End}(W)$ be such that $P \circ a = P \circ b$. For any $v \in \ker(b)$, the chain rule yields

$$\nabla(P \circ a) \cdot v = \nabla(P \circ b) \cdot v = (\nabla(P) \circ b) \cdot b \cdot v = 0,$$

so any $v \in \ker(b)$ is a linear relation among the partial derivatives of $P \circ a$. In other words, $k = \dim(\ker(\delta_a)) \geq \dim(\ker(b))$. Therefore,

$$\operatorname{rk}(a) \leq r = n - k \leq n - \dim(\ker(b)) = \operatorname{rk}(b).$$ ■

5.2 Grading of Coordinate Rings and Projectivization

The next observation is that Ω_P and $\overline{\Omega}_P$ are both *affine cones* in the following sense:

5.2.1 Definition. A subset $C \subseteq V$ is called an **affine cone** if $\forall P \in C\colon \mathbb{C} \cdot P \subseteq C$. ●

Note that $\mathrm{GL}(W)$ contains a central copy of $\mathbb{G}_{\mathrm{m}}(\mathbb{C}) = \mathbb{C}^\times$ in the form of scalar maps $\mathbb{C}^\times \cdot \mathrm{id}_W$, where id_W denotes the identity map. Given $Q \in \Omega_P \subseteq \mathbb{C}[W]_d$ and any $t \in \mathbb{C}$, let $\zeta \in \mathbb{C}$ be some d-th root of t. Then, $tQ = \zeta^d Q = Q \circ \zeta\,\mathrm{id}_W \in \Omega_Q = \Omega_P$. Since scalar multiplication is continuous, we have the following general principle.

5.2.2 Lemma. If $C \subseteq V$ is an affine cone, then $\overline{C}$ is also an affine cone.

Proof. The reductive group $\mathbb{G}_{\mathrm{m}} = (\mathbb{C}^\times, \cdot)$ acts algebraically on V by scalar multiplication. Denote by $\alpha\colon \mathbb{G}_{\mathrm{m}} \times V \to V$ the action morphism. Since $\overline{\mathbb{G}_{\mathrm{m}} \times C} = \mathbb{G}_{\mathrm{m}} \times \overline{C}$, we have $\alpha(\mathbb{G}_{\mathrm{m}} \times \overline{C}) \subseteq \overline{\alpha(\mathbb{G}_{\mathrm{m}} \times C)} = \overline{C}$. Hence, $\overline{C}$ is $\mathbb{G}_{\mathrm{m}}$-invariant. ∎

This means that $\overline{\Omega}_P$ is an affine cone. In particular, the coordinate ring $\mathbb{C}[\overline{\Omega}_P]$ inherits a $\mathbb{N}$-grading from $\mathbb{C}[V]$. On the other hand, $\mathbb{C}[\mathrm{GL}_n]$ has a natural $\mathbb{Z}$-grading that is the result of localizing at the SL_n-invariant polynomial $\det_n$. The coordinate ring $\mathbb{C}[\Omega_P] = \mathbb{C}[\mathrm{GL}_n]^{G_P}$ inherits this grading. By Proposition A.2.9,

$$\mathbb{C}[\Omega_P]_d = \bigoplus_{\substack{\lambda \in \Lambda_n^+ \\ |\lambda| = d}} \mathbb{V}(\lambda)^* \otimes \mathbb{V}(\lambda)^{G_P}.$$

We will now see that under reasonable assumptions, $\mathbb{C}[\Omega_P]$ is also the localization of $\mathbb{C}[\overline{\Omega}_P]$ at some $\mathrm{SL}(W)$-invariant and the grading of $\mathbb{C}[\overline{\Omega}_P]$ is the same as the natural one induced by this localization. We call a form $P \in \mathbb{C}[W]_d$ **polystable** if $P \circ \mathrm{SL}(W)$ is Zariski closed in $\mathbb{C}[W]_d$. Both the permanent and the determinant are polystable [BI15, 2.9]. Furthermore, Bürgisser and Ikenmeyer proved:

5.2.3 Theorem ([BI15, 3.9]). If $P \in V = \mathbb{C}[W]_d$ is polystable, then there exists a homogeneous invariant $f \in \mathbb{C}[\overline{\Omega}_P]^{\mathrm{SL}(W)}$ such that $\mathbb{C}[\Omega_P] = \mathbb{C}[\overline{\Omega}_P][f^{-1}]$. In particular,

$$\Omega_P = \left\{ Q \in \overline{\Omega}_P \;\middle|\; f(Q) \neq 0 \right\}.$$

Furthermore, if f is homogeneous of minimal degree with this property, then f is unique up to scalar. ∎

Remark. Bürgisser and Ikenmeyer also study the minimal degree and other numerical quantities related to these invariants, for the determinant and several other interesting families of polynomials. ○

Studying the geometry of $\overline{\Omega}_P$ is equivalent to studying its projectivization $\mathbb{P}(\overline{\Omega}_P)$, which is a projective variety. Furthermore, we know that $\mathbb{P}(\Omega_P)$ is a smooth, open subset of $\mathbb{P}(\overline{\Omega}_P)$ which we understand well. While this is quite promising, it turns out [BI15, 3.10,3.17,3.29] that $\overline{\Omega}_P$ is not a normal variety in most cases and in particular, when P is $\det_n$ or per_n with $n \geq 3$. It follows that in all these cases, $\overline{\Omega}_P$ has singularities. Since Ω_P is smooth, these singularities will be part of the **boundary** $\partial\Omega_P := \overline{\Omega}_P \setminus \Omega_P$, which is therefore a subvariety that needs to be analyzed further. We note first that it is a hypersurface in our cases of interest:

5.2.4 Lemma. If G_P is reductive, then $\partial\Omega_P$ is pure of codimension one in $\overline{\Omega}_P$.

Proof. The complement of an affine variety is always pure of codimension one [Gro67, Corollaire 21.12.7]. The statement follows because for reductive G_P, the orbit Ω_P is affine by Theorem A.1.9. ■

5.3 Rational Orbit Map

The orbit map $\omega_P \colon \operatorname{End}(W) \to \overline{\Omega}_P$ given by $a \mapsto P \circ a$ is line-preserving, therefore it can be viewed as a rational map

$$\begin{aligned} \varpi_P \colon \mathbb{P}\operatorname{End}(W) &\dashrightarrow \mathbb{P}\overline{\Omega}_P \\ [a] &\longmapsto [P \circ a] \end{aligned} \tag{1}$$

Since it can happen that $P \circ a = 0$ for $a \neq 0$, the map ϖ_P is not defined everywhere. If it was defined everywhere, it would be a dominant, projective morphism, therefore surjective [Sha94, I.5.2]. We will transform ϖ_P into a projective morphism by two steps, the first of which is explained in this section. In Chapter 6 we study the special case where we obtain a morphism after the first step already. The second step is postponed until Section 7.3 and we will see how to combine both steps in the final two chapters.

5.3.1 Definition. For a form $P \in \mathbb{C}[W]_d$, we define

$$\mathcal{A}_P := \{a \in \operatorname{End}(W) \mid P \circ a = 0\} \tag{2}$$

and call it the **annihilator** of P. It is an affine cone in $\operatorname{End}(W)$ whose projectivization is precisely the set of points where $\varpi_P \colon \mathbb{P}\operatorname{End}(W) \dashrightarrow \mathbb{P}\overline{\Omega}_P$ is not defined. ●

5.3.2 Remark. Another way to see the annihilator is as

$$\mathcal{A}_P = \{a \in \operatorname{End}(W) \mid a(W) \subseteq \operatorname{Z}(P)\},$$

the set of all endomorphisms whose image is a linear subspace of the hypersurface $\mathrm{Z}(P) \subseteq W$. We will call a linear subspace $L \subseteq \mathrm{Z}(P)$ **maximal** if there is no other linear subspace of $\mathrm{Z}(P)$ which properly contains L. With this notion, $\mathcal{A}_P$ is the set of all $a \in \mathrm{End}(W)$ whose image is contained in some maximal linear subspace of $\mathrm{Z}(P)$. ●

The stabilizer $G_P = \{g \in \mathrm{GL}(W) \mid P \circ g = P\}$ acts by left multiplication on $\mathrm{End}(W)$ and ω_P is G_P-invariant with respect to this action, since $\omega_P(g \circ a) = P \circ g \circ a = P \circ a$ for all $g \in G_P$. This action induces an action on $\mathbb{P}\,\mathrm{End}(W)$ and consequently, the rational map ϖ_P from (1) is invariant with respect to this induced action. We stress again that G_P is assumed reductive unless explicitly stated otherwise.

Consider now a map $a \in \mathrm{End}(W)$ with $0 \in \overline{G_P \circ a}$, i.e., there is a sequence $(g_n)_{n\in\mathbb{N}}$ with $g_n \in G_P$ such that $\lim_{n\to\infty} g_n \circ a = 0$. Such points $a \in \mathrm{End}(W)$ are called **unstable** with respect to the action of G_P in the sense of geometric invariant theory [MFK94], see also [New78; Kra85; Dol03].

5.3.3 Definition. Let P be a homogeneous form. The set of unstable endomorphisms with respect to the action of G_P is called the **nullcone** of P and we denote it by

$$\mathcal{N}_P := \left\{ a \in \mathrm{End}(W) \;\middle|\; 0 \in \overline{G_P \circ a} \right\}. \tag{3}$$

See Subsection A.1.2 for other, equivalent definitions of the nullcone. ●

A central observation is the following:

5.3.4 Proposition. We have $\mathcal{N}_P \subseteq \mathcal{A}_P$. In other words, if $a \in \mathrm{End}(W)$ is unstable with respect to the left action of G_P, then $P \circ a = 0$. This statement holds regardless of whether or not G_P is reductive.

Proof. Let $a \in \mathcal{N}_P$, meaning $0 \in \overline{G_P \circ a}$. Since $\omega_P^{-1}(P \circ a)$ is a closed subset of $\mathrm{End}(W)$ containing $G_P \circ a$, it also contains its closure. Thus, $0 \in \overline{G_P \circ a} \subseteq \omega_P^{-1}(P \circ a)$, which means $P \circ a = P \circ 0 = 0$. ∎

We will now explain that the points in $\mathcal{N}_P$ induce "harmless" indeterminacies of ϖ_P. We require the results from Subsection A.1.2, in particular the notion of semistability. In our situation,

$$\mathrm{End}(W)^{\mathrm{ss}} := \mathrm{End}(W) \setminus \mathcal{N}_P = \left\{ a \in \mathrm{End}(W) \;\middle|\; 0 \notin \overline{G_P \circ a} \right\}$$

is the set of **semistable** endomorphisms and $\mathbb{P}\,\mathrm{End}(W)^{\mathrm{ss}} \subseteq \mathbb{P}\,\mathrm{End}(W)$ is an open subvariety of $\mathbb{P}\,\mathrm{End}(W)$ which admits a categorical quotient, see Proposition A.1.11. An important property of this quotient is the fact that

$$\mathbb{P}\,\mathrm{End}(W)^{\mathrm{ss}} /\!\!/ G_P = \mathrm{Proj}(\mathbb{C}[\mathrm{End}(W)]^{G_P})$$

is a projective variety, even though $\mathbb{P}\operatorname{End}(W)^{\mathrm{ss}}$ is only quasi-projective in general.

The first step for transforming ϖ_P into a morphism is the following proposition, which follows immediately from Propositions A.1.12 and 5.3.4.

5.3.5 Proposition. The domain of definition of $\varpi_P \colon \mathbb{P}\operatorname{End}(W) \dashrightarrow \mathbb{P}\overline{\Omega}_P$ is contained in the open subset $\mathbb{P}\operatorname{End}(W)^{\mathrm{ss}}$. Furthermore, there is a commutative diagram

$$\begin{tikzcd}
\mathbb{P}\operatorname{End}(W)^{\mathrm{ss}} \arrow[r, dashed, "\varpi_P"] \arrow[d, two heads, "\pi"'] & \mathbb{P}\overline{\Omega}_P \\
\mathbb{P}\operatorname{End}(W)^{\mathrm{ss}} /\!\!/ G_P \arrow[ur, dashed, "\phi_P"'] &
\end{tikzcd}$$

and $\mathbb{P}\operatorname{End}(W)^{\mathrm{ss}} /\!\!/ G_P$ is a projective variety. ∎

Remark. We can think of the rational map ϕ_P as having less indeterminacy than ϖ_P because $\mathcal{N}_P$ has been removed before passing to the quotient. ○

For $P \in \mathbb{C}[W]_d$, the stabilizer $G_P \subseteq \mathrm{GL}(W)$ acts on the vanishing set

$$\mathrm{Z}(P) := \{ w \in W \mid P(w) = 0 \} \subseteq W,$$

because $P(w) = 0$ implies $P(g(w)) = (P \circ g)(w) = P(z) = 0$ for any $g \in G_P$. In particular, G_P acts on the set of linear subspaces of $\mathrm{Z}(P)$. We will often study this action, in particular to determine the structure of G_P in some cases. In light of Remark 5.3.2, we therefore introduce the following terminology:

5.3.6 Definition. For $a \in \operatorname{End}(W)$, we denote by $\operatorname{im}(a) := a(W)$ its **image**. A linear subspace $L \subseteq \mathrm{Z}(P)$ is called **semistable** if there is some $a \in \operatorname{End}(W)^{\mathrm{ss}}$ with $L = \operatorname{im}(a)$. Otherwise, L is called **unstable**. ●

5.3.7 Remark.

(1) For any $a \in \mathcal{N}_P$, the space $L := \operatorname{im}(a)$ is unstable: If some $b \in \operatorname{End}(W)$ satisfies $L = \operatorname{im}(b)$, then there is a $g \in \mathrm{GL}(W)$ with $b = a \circ g$ and since g is an automorphism of $\operatorname{End}(W)$, we have $0 \in \overline{G_P \circ a} \circ g = \overline{G_P \circ a \circ g} = \overline{G_P \circ b}$.

(2) Subspaces of unstable linear subspaces are unstable. Indeed, let $a \in \mathcal{N}_P$ and $L \subseteq \operatorname{im}(a)$ a subspace. Then, $L = \operatorname{im}(a \circ p)$, where $p \in \operatorname{End}(W)$ is some projection. Since $0 \in \overline{G_P \circ a} \circ p \subseteq \overline{G_P \circ a \circ p}$, the space L is also unstable. ●

Chapter 6

Closed Forms

If $P \in \mathbb{C}[W]_d$ satisfies $\overline{\Omega}_P = P \circ \operatorname{End}(W)$, we call it **closed**. This is equivalent to the fact that ω_P has closed image.

6.0.1 Remark. If P is closed, then $\partial\Omega_P$ is always irreducible: It is the image under $\omega_P\colon \operatorname{End}(W) \to \overline{\Omega}_P$ of the irreducible hypersurface $\mathcal{S} \subseteq \operatorname{End}(W)$ of noninvertible endomorphisms. ●

6.0.2 Example. Every quadratic form is closed. In fact, if there is some symmetric matrix $b \in \mathbb{C}^{n\times n}$ such that $P(x) = x^{\mathrm{t}}bx$, then for any endomorphism a, the form $P \circ a$ is given by the matrix $a^{\mathrm{t}}ba$. We may assume that P is concise, i.e. b is invertible.

It follows that $\{a^{\mathrm{t}}ba \mid a \in \mathbb{C}^{n\times n}\} = \{a^{\mathrm{t}}a \mid a \in \mathbb{C}^{n\times n}\}$ is the space of symmetric matrices, hence $P \circ \operatorname{End}(\mathbb{C}^n) = \mathbb{C}[x_1, \ldots, x_n]_2$. In particular, this is equal to $\overline{\Omega}_P$ and P is closed. ●

6.1 A Sufficient Criterion

We know only very few examples of closed forms of higher degree. A classification remains an interesting open problem, we can only give some partial results and pose several questions. Recall the definitions of $\mathcal{A}_P$ and $\mathcal{N}_P$ from Definitions 5.3.1 and 5.3.3. A sufficient criterion for being closed is the following:

6.1.1 Proposition. Let $P \in \mathbb{C}[W]_d$ be a form with reductive stabilizer $G_P \subseteq \operatorname{GL}(W)$. If we have $\mathcal{A}_P = \mathcal{N}_P$, then P is closed.

Proof. Recall the definition of ϖ_P from (1) and Proposition 5.3.5. The assumption means that ϖ_P is defined everywhere on $\mathbb{P}\operatorname{End}(W)^{\mathrm{ss}}$ and so the induced rational map $\phi_P\colon \mathbb{P}\operatorname{End}(W)^{\mathrm{ss}} /\!\!/ G_P \to \mathbb{P}\overline{\Omega}_P$ is in fact a morphism. Since it is dominant and projective, its image is equal to $\mathbb{P}\overline{\Omega}_P$. The statement follows because ϖ_P has the same image as ϕ_P. ■

6.1.2 Corollary. If the hypersurface $\mathrm{Z}(P)$ has no semistable linear subspaces, then P is closed. ■

In Proposition 5.3.4, we saw $\mathcal{N}_P \subseteq \mathcal{A}_P$. Proposition 6.1.1 states that P is closed if this inclusion is an equality. Having no counterexample but also no wide selection of examples in general, we ask the following question:

6.1.3 Question. Does $\mathcal{N}_P = \mathcal{A}_P$ hold for every closed form P?

We know of no counterexample for Question 6.1.3, but for two examples of closed forms that are already well-known, we illustrate that $\mathcal{N}_P = \mathcal{A}_P$.

6.1.4 Example. We know from Example 6.0.2 that concise quadratic forms are closed. We claim that $\mathcal{N}_P = \mathcal{A}_P$ for any such $P \in V = \mathbb{C}[x_1, \ldots, x_n]_2$. We may assume without loss of generality that $P = x_1^2 + \cdots + x_n^2$ because any concise quadratic form is an element of Ω_P. On the one hand, we observe that

$$\mathcal{A}_P = \{a \in \mathbb{C}^{n\times n} \mid P \circ a = 0\} = \{a \in \mathbb{C}^{n\times n} \mid a^{\mathrm{t}}a = 0\}.$$

The stabilizer of P is the orthogonal group $\mathrm{O}_n = \{a \in \mathrm{GL}_n \mid a^{\mathrm{t}} = a^{-1}\}$. As before, we consider the operation of O_n on $\mathbb{C}^{n\times n}$ by left multiplication. By the First Fundamental Theorem for the Orthogonal Group [Pro06, 11.2, Theorem on p. 390], the O_n-invariant functions on $\mathbb{C}^{n\times n}$ are generated by the entries of the map $a \mapsto a^{\mathrm{t}}a$, and by Lemma A.1.10 we therefore also have $\mathcal{N}_P = \{a \in \mathbb{C}^{n\times n} \mid a^{\mathrm{t}}a = 0\}$. ●

6.1.5 Example. Consider $P := z(y^2 + xz) \in \mathbb{C}[x, y, z]_3$. The classification of ternary cubic forms [KM02] implies that P is closed. We will show that $\mathcal{A}_P = \mathcal{N}_P$. By Example 2 in said reference, the stabilizer $G_P \subseteq \mathrm{GL}_3$ is generated by the matrices

$$u_\alpha := \begin{pmatrix} 1 & -2\alpha & -\alpha^2 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad t_\varepsilon := \begin{pmatrix} \varepsilon^4 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^{-2} \end{pmatrix}.$$

Note that $U := \{u_\alpha \mid \alpha \in \mathbb{C}\} \cong (\mathbb{C}, +)$ is a normal unipotent subgroup of G_P while the group $T := \{t_\varepsilon \mid \varepsilon \in \mathbb{C}^\times\} \cong (\mathbb{C}^\times, \cdot)$ is reductive. More precisely, $t_\varepsilon u_\alpha t_\varepsilon{}^{-1} = u_{\varepsilon^3\alpha}$. We have $G_P = T \ltimes U$, the unipotent radical of G_P is equal to U and in particular G_P is not reductive and we cannot appeal to Proposition 6.1.1 to show that it is closed. However, the definitions of $\mathcal{N}_P$ and $\mathcal{A}_P$ do not require P to have reductive stabilizer.

Observe that $\mathrm{Z}(P) = \mathrm{Z}(z) \cup \mathrm{Z}(y^2 + xz)$. The maximal linear subspaces of $\mathrm{Z}(P)$ are therefore the plane $\mathrm{Z}(z)$ and any line in $\mathrm{Z}(y^2 + xz)$. We first note that $\mathrm{Z}(z)$ is unstable because for any $w \in \mathrm{Z}(z)$, we have $t_\varepsilon(w) \to 0$ as $\varepsilon \to 0$. Therefore, we are left to show that any $w \in \mathrm{Z}(y^2 + xz) \setminus \mathrm{Z}(z)$ spans an unstable line. After applying an appropriate scaling matrix t_ε, we can assume that $w = (-y^2, y, 1)$. Consider

$$g_\varepsilon := t_{\varepsilon^{-1}} \circ u_{\varepsilon^3 - y} = \frac{1}{\varepsilon^4} \cdot \begin{pmatrix} 1 & -2(\varepsilon^3 - y) & -(\varepsilon^3 - y)^2 \\ 0 & \varepsilon^3 & \varepsilon^3(\varepsilon^3 - y) \\ 0 & 0 & \varepsilon^6 \end{pmatrix}.$$

A straightforward computation shows that $g_\varepsilon(w) = (-\varepsilon^2, \varepsilon^2, \varepsilon^2) \to 0$ as $\varepsilon \to 0$. ●

6.2 Normalizations of Orbit Closures

For almost every family of homogeneous forms studied in the context of GCT, the orbit closures are not normal varieties [Kum15; BI11]. In Section 4.3, we saw a prominent example of an orbit closure whose normalization has a nice representation-theoretic description: For this section, we choose coordinates $\mathbb{C}[W] = \mathbb{C}[x_1, \ldots, x_n]$ and consider the n-form $\mathrm{mn}_n := x_1 \cdots x_n$, the universal monomial. It is also the top symmetric form. The variety $\mathscr{C}_n = \overline{\Omega}(\mathrm{mn}_n)$ is called the n-th Chow variety. In Chapter 4 we also used the notation $\mathscr{D}_n = \overline{\Omega}(\det_n)$, and we recall that $\mathscr{C}_n \subseteq \mathscr{D}_n$ because mn_n is the restriction of $\det_n$ to diagonal matrices.

By Lemma 4.3.1, the universal monomial is closed. Again, we observe that it actually satisfies the condition we discussed in the previous section:

6.2.1 Proposition. We have $\mathcal{N}_{\mathrm{mn}_n} = \mathcal{A}_{\mathrm{mn}_n}$ for all $n \geq 2$.

Proof. We fix $n \geq 2$ and let $P := \mathrm{mn}_n$. We show that any $a \in \mathrm{End}(W)$ with $P \circ a = 0$ satisfies $0 \in \overline{G_P \circ a}$. The group G_P contains all diagonal matrices $t = \mathrm{diag}(t_1, \ldots, t_n)$ with $1 = t_1 \cdots t_n$. Assume that $a \in \mathbb{C}^{n \times n} \cong \mathrm{End}(W)$ satisfies $P \circ a = 0$ and let $a_i := x_i \circ a = \sum_{j=1}^{n} a_{ij} x_j$, then we have $0 = P \circ a = a_1 \cdots a_n$. Since $\mathbb{C}[W]$ is an integral domain, it follows that there must be some i with $a_i = 0$. Without loss of generality, we may assume that $a_1 = 0$, i.e., the first row of a contains only zeros. The diagonal matrix $t_\varepsilon := \mathrm{diag}(\varepsilon^{1-n}, \varepsilon, \ldots, \varepsilon)$ stabilizes P and $t_\varepsilon \circ a = \varepsilon \cdot a$. We can view t_ε as a regular map $t \colon \mathbb{C} \to \mathrm{End}(W)$ mapping $\varepsilon \mapsto t_\varepsilon$. Since $t(\mathbb{C}^\times) \subseteq G_P$, we have $t(\mathbb{C}^\times) \circ a \subseteq G_P \circ a$. Passing to the closure, we get $0 = t_0 \circ a \in \overline{G_P \circ a}$. ∎

The following theorem is well-known. It is implied by the proof of Lemma 4.3.3, but we will also obtain it as a corollary of the upcoming Theorem 6.2.3 – we only quote it here to provide context. Refer to Section A.3 for the notion of the polynomial part $V_{\sqsupseteq 0}$ of a $\mathrm{GL}(W)$-module V.

6.2.2 Theorem. Let $\nu \colon \mathrm{N}(\mathscr{C}_n) \to \mathscr{C}_n$ be the normalization of the Chow variety. The morphism ν corresponds to the inclusion $\mathbb{C}[\mathrm{N}(\mathscr{C}_n)] = \mathbb{C}[\mathscr{C}_n]_{\sqsupseteq 0} \subseteq \mathbb{C}[\mathscr{C}_n]$. ∎

Arguably, Theorem 6.2.2 and Remark 6.0.1 motivated Landsberg [Lan15] to ask the following question: Is it true that whenever a $\mathrm{GL}(W)$-orbit closure with reductive stabilizer has an irreducible boundary, the coordinate ring of the normalization of the orbit closure equals the polynomial part of the coordinate ring of the orbit?

The answer is negative, as we recently showed [Hü17]. We will instead prove that $\mathbb{C}[\mathrm{N}(\overline{\Omega}_P)] = \mathbb{C}[\overline{\Omega}_P]_{\sqsupseteq 0}$ is equivalent to $\mathcal{A}_P = \mathcal{N}_P$. Furthermore, we illustrate that these equivalent conditions can be violated in cases where $\partial\Omega_P$ is irreducible. The main theorem of this chapter is the following, its proof is postponed to Section 6.3.

6.2.3 Theorem. Let V be a polynomial right-$\mathrm{GL}(W)$-module where we denote the action by $V \times \mathrm{GL}(W) \to V$, $(P, g) \mapsto P \circ g$. Let $P \in V$ be a point with reductive stabilizer $G \subseteq \mathrm{GL}(W)$. Denote by $\Omega = P \circ \mathrm{GL}(W)$ its orbit and by $\nu\colon \mathrm{N}(\overline{\Omega}) \to \overline{\Omega}$ the normalization of its orbit closure.

There is a canonical, injective homomorphism $\iota\colon \mathbb{C}[\mathrm{N}(\overline{\Omega})] \hookrightarrow \mathbb{C}[\Omega]_{\sqsupseteq 0}$ of graded $\mathbb{C}$-algebras and $\mathrm{GL}(W)$-modules, and the following statements are equivalent:

(1) The injection ι is an isomorphism.

(2) For all $a \in \mathrm{End}(W)$ with $P \circ a = 0$, we have $0 \in \overline{G \circ a}$.

(This means $\mathcal{A}_P = \mathcal{N}_P$ in the case of homogeneous forms)

If either condition is satisfied, we have $P \circ \mathrm{End}(W) = \overline{\Omega}$.

Note that Theorem 6.2.3 implies Theorem 6.2.2 by Proposition 6.2.1. Furthermore, we can use Theorem 6.2.3 to quickly deduce that the orbit of an elliptic curve is a counterexample for the original question by Landsberg:

6.2.4 Proposition. Let $W = \mathbb{C}^3$ and let $P \in \mathbb{C}[W]_3$ be any form that defines a non-singular curve in $\mathbb{P}^2$. Let $\nu\colon \mathrm{N}(\overline{\Omega}_P) \to \overline{\Omega}_P$ be the normalization of $\overline{\Omega}_P$. Then, the stabilizer of P is reductive, $\partial\Omega_P$ is irreducible and $\mathbb{C}[\mathrm{N}(\overline{\Omega}_P)]$ is not isomorphic to $\mathbb{C}[\Omega_P]_{\sqsupseteq 0}$ as $\mathrm{GL}(W)$-modules.

For the proof, we require some classical results about ternary cubics to deduce that the stabilizer of P is reductive and the boundary of its orbit irreducible:

6.2.5 Lemma. If $P \in \mathbb{C}[W]_3$ defines a smooth curve, it has a finite (hence reductive) stabilizer and $\partial\Omega_P = \overline{\Omega}_Q$ is the orbit closure of a concise form Q. In particular, $\partial\Omega_P$ is irreducible.

Proof. By [KM02, Corollary 1], the stabilizer of P is finite. Any finite group is reductive by Maschke's Theorem. A complete diagram of the degeneracy behaviour of all ternary cubic forms can be found in Section 4 of [KM02]. Choosing coordinates $\mathbb{C}[W] = \mathbb{C}[x, y, z]$, the diagram implies that $\partial\Omega_P$ is equal to the orbit closure of the polynomial $Q := x^3 - y^2 z$, regardless of the choice of P. One can compute that Q is concise by means of Corollary 5.1.4. Hence, $\partial\Omega_P = \overline{\Omega}_Q$. This variety is irreducible because it is the closure of the image of the irreducible variety $\mathrm{GL}(W)$ under the regular map $\mathrm{GL}(W) \to \mathbb{C}[W]_3$, $g \mapsto Q \circ g$. ∎

Proof of Proposition 6.2.4. By Lemma 6.2.5, the polynomial P has a reductive stabilizer and its orbit has an irreducible boundary. Let $[w] \in \mathbb{P}^2$ be any point on the curve defined by P, i.e., $w \in W$ is nonzero and $P(w) = 0$. Let $a \in \mathrm{End}(W)$ be of rank one such that $\mathrm{im}(a)$ is spanned by w. Then, $P \circ a = 0$ and since G_P is a finite group, $\overline{G_P \circ a} = G_P \circ a$ does not contain the zero map. Hence by Theorem 6.2.3, the two $\mathrm{GL}(W)$-modules $\mathbb{C}[\mathrm{N}(\overline{\Omega}_P)]$ and $\mathbb{C}[\Omega_P]_{\sqsupseteq 0}$ are not isomorphic. ∎

This counterexample also yields another important observation:

6.2.6 Corollary. Let $\mathcal{S} := \mathrm{End}(W) \setminus \mathrm{GL}(W)$ be the hypersurface of noninvertible endomorphisms. Then, $\overline{\omega_P(\mathcal{S})} \subseteq \partial\Omega_P$ is not an irreducible component of $\partial\Omega_P$ in general.

Proof. With P as in Lemma 6.2.5, we know that Zariski almost every element of $\partial\Omega_P$ is concise, so the image of $\omega_P(\mathcal{S})$ cannot be dense in $\partial\Omega_P$. ∎

6.2.1 The Aronhold Hypersurface

As an example for Proposition 6.2.4, we consider the special case of the Fermat cubic: Let $W = \mathbb{C}^3$, $V = \mathbb{C}[W]_3 = \mathbb{C}[x,y,z]_3$ and $P := x^3 + y^3 + z^3 \in V$ for the rest of this subsection. We will write Ω and $\overline{\Omega}$ instead of Ω_P and $\overline{\Omega}_P$ to simplify notation. By Theorem 6.2.3 and Proposition 6.2.4, we know that the quotient $\mathbb{C}[\Omega]_{\sqsupseteq 0}/\mathbb{C}[\mathrm{N}(\overline{\Omega})]$ exists and that it is a nontrivial GL_3-module, so it decomposes as a direct sum of irreducible GL_3-modules. We will explicitly compute some of the corresponding multiplicities.

Note that this is a special case as the orbit closure $\overline{\Omega} \subseteq V$ is a normal variety. Indeed, this follows because $\overline{\Omega}$ is the third secant variety of the 3-uple veronese embedding $\mathbb{P}^1 \to \mathbb{P}^3$, which is projectively normal by [IK99, Thm. 1.56]. This simplifies the calculation because we need not determine the normalization of $\overline{\Omega}$. Note that if P is a generic regular cubic, $\overline{\Omega}$ is not normal [BI15, Cor. 3.17 (1)].

P defines the elliptic curve with j-invariant equal to zero [Stu93, 4.4.7,4.5.8]. Its orbit closure $\overline{\Omega}$ is the hypersurface defined by the Aronhold invariant $\alpha \in \mathbb{C}[V]_4$, see [BI15, §3.2.1] for an explicit description. Thus, $\mathbb{C}[\mathrm{N}(\overline{\Omega})] = \mathbb{C}[\overline{\Omega}] = \mathbb{C}[V]/\langle\alpha\rangle$. We write in general

$$\mathrm{Sym}^d \mathrm{Sym}^m \mathbb{C}^3 = \bigoplus_{\lambda\in\Lambda} \mathbb{V}(\lambda)^{\mathrm{p}(d,m;\lambda)},$$

the coefficients $\mathrm{p}(d,m;\lambda)$ are also known as plethysm coefficients. In this case, we have $m = 3$.

The Aaronhold invariant α is a degree 4 polynomial and up to scaling, the unique highest weight vector in $\mathbb{C}[V]$ of weight $(4,4,4)$ with respect to the action of $\mathrm{GL}(W) \cong \mathrm{GL}_3(\mathbb{C})$. This means that the linear span of the GL_3-orbit of α is isomorphic to the irreducible GL_3-module $\mathbb{V}((4,4,4))$. We denote by $\Lambda := \Lambda_3^+$ the set of dominant weights of $\mathrm{GL}_3(\mathbb{C})$, see Paragraph A.2.8. The degree 4 part of the homogeneous ideal generated by α then decomposes as

$$\langle\alpha\rangle_d = \mathbb{C}[V]_{d-4}\cdot\alpha \cong \bigoplus_{\lambda\in\Lambda} \mathbb{V}(\lambda)^{\mathrm{p}(d-4,3;\lambda-(4,4,4))}$$

and we get

$$\mathbb{C}[\overline{\Omega}]_d \cong \bigoplus\nolimits_{\lambda\in\Lambda} \mathbb{V}(\lambda)^{b_\lambda} \quad \text{where} \quad b_\lambda = \mathrm{p}(d,3;\lambda) - \mathrm{p}(d-4,3;\lambda-(4,4,4)).$$

a_λ	b_λ	m_λ	λ	a_λ	b_λ	m_λ	λ	a_λ	b_λ	m_λ	λ	a_λ	b_λ	m_λ	λ
1	1	0	(4,2,0)	1	1	0	(8,8,2)	4	3	1	(12,6,3)	2	2	0	(13,7,4)
2	1	1	(6,0,0)	2	2	0	(9,6,3)	4	2	2	(12,7,2)	4	3	1	(13,8,3)
1	1	0	(4,4,1)	1	1	0	(9,7,2)	5	1	4	(12,8,1)	5	2	3	(13,9,2)
1	1	0	(5,2,2)	1	0	1	(9,8,1)	6	1	5	(12,9,0)	5	1	4	(13,10,1)
2	1	1	(6,3,0)	1	0	1	(9,9,0)	1	1	0	(13,4,4)	4	0	4	(13,11,0)
2	1	1	(7,2,0)	1	1	0	(10,4,4)	2	1	1	(13,5,3)	4	3	1	(14,6,4)
1	0	1	(8,1,0)	1	1	0	(10,5,3)	6	3	3	(13,6,2)	4	2	2	(14,7,3)
3	1	2	(9,0,0)	3	2	1	(10,6,2)	5	1	4	(13,7,1)	7	3	4	(14,8,2)
1	1	0	(6,4,2)	3	1	2	(10,7,1)	8	1	7	(13,8,0)	7	1	6	(14,9,1)
2	1	1	(6,6,0)	4	1	3	(10,8,0)	3	1	2	(14,4,3)	9	1	8	(14,10,0)
1	1	0	(7,3,2)	1	1	0	(11,4,3)	5	2	3	(14,5,2)	2	1	1	(15,5,4)
1	1	0	(7,4,1)	3	2	1	(11,5,2)	8	1	7	(14,6,1)	6	3	3	(15,6,3)
1	0	1	(7,5,0)	4	1	3	(11,6,1)	9	1	8	(14,7,0)	7	3	4	(15,7,2)
1	1	0	(8,2,2)	3	0	3	(11,7,0)	1	0	1	(15,3,3)	9	1	8	(15,8,1)
1	0	1	(8,3,1)	1	0	1	(12,3,3)	6	2	4	(15,4,2)	11	1	10	(15,9,0)
3	1	2	(8,4,0)	4	2	2	(12,4,2)	6	0	6	(15,5,1)	2	1	1	(16,4,4)
1	0	1	(9,2,1)	3	0	3	(12,5,1)	13	2	11	(15,6,0)	4	1	3	(16,5,3)
3	1	2	(9,3,0)	9	2	7	(12,6,0)	5	1	4	(16,3,2)	10	3	7	(16,6,2)
4	1	3	(10,2,0)	3	1	2	(13,3,2)	8	1	7	(16,4,1)	10	1	9	(16,7,1)
2	0	2	(11,1,0)	5	1	4	(13,4,1)	12	1	11	(16,5,0)	15	2	13	(16,8,0)
4	1	3	(12,0,0)	7	1	6	(13,5,0)	4	1	3	(17,2,2)	4	1	3	(17,4,3)
1	1	0	(6,6,3)	3	1	2	(14,2,2)	6	0	6	(17,3,1)	8	2	6	(17,5,2)
1	1	0	(7,6,2)	4	0	4	(14,3,1)	13	1	12	(17,4,0)	12	1	11	(17,6,1)
1	1	0	(8,4,3)	9	1	8	(14,4,0)	5	0	5	(18,2,1)	14	1	13	(17,7,0)
1	1	0	(8,5,2)	3	0	3	(15,2,1)	13	1	12	(18,3,0)	2	0	2	(18,3,3)
2	1	1	(8,6,1)	9	1	8	(15,3,0)	1	0	1	(19,1,1)	9	2	7	(18,4,2)
1	0	1	(8,7,0)	1	0	1	(16,1,1)	12	1	11	(19,2,0)	10	0	10	(18,5,1)
2	2	0	(9,4,2)	9	1	8	(16,2,0)	8	0	8	(20,1,0)	20	2	18	(18,6,0)
1	0	1	(9,5,1)	5	0	5	(17,1,0)	8	1	7	(21,0,0)	7	1	6	(19,3,2)
4	1	3	(9,6,0)	7	1	6	(18,0,0)	1	1	0	(10,8,6)	11	1	10	(19,4,1)
2	1	1	(10,3,2)	1	1	0	(9,6,6)	1	1	0	(10,9,5)	17	1	16	(19,5,0)
3	1	2	(10,4,1)	1	1	0	(9,8,4)	1	1	0	(10,10,4)	5	1	4	(20,2,2)
4	1	3	(10,5,0)	1	1	0	(10,6,5)	1	1	0	(11,8,5)	9	0	9	(20,3,1)
2	1	1	(11,2,2)	1	1	0	(10,7,4)	1	1	0	(11,9,4)	19	1	18	(20,4,0)
2	0	2	(11,3,1)	2	2	0	(10,8,3)	1	1	0	(11,10,3)	7	0	7	(21,2,1)
5	1	4	(11,4,0)	2	1	1	(10,9,2)	2	2	0	(12,6,6)	17	1	16	(21,3,0)
2	0	2	(12,2,1)	2	1	1	(10,10,1)	1	1	0	(12,7,5)	2	0	2	(22,1,1)
6	1	5	(12,3,0)	2	2	0	(11,6,4)	3	3	0	(12,8,4)	16	1	15	(22,2,0)
6	1	5	(13,2,0)	1	1	0	(11,7,3)	3	2	1	(12,9,3)	10	0	10	(23,1,0)
4	0	4	(14,1,0)	3	2	1	(11,8,2)	4	2	2	(12,10,2)	10	1	9	(24,0,0)
5	1	4	(15,0,0)	2	0	2	(11,9,1)	2	0	2	(12,11,1)	....			
1	1	0	(6,6,6)	2	0	2	(11,10,0)	4	1	3	(12,12,0)				
1	1	0	(8,6,4)	1	1	0	(12,5,4)	2	2	0	(13,6,5)				

Figure 6.2.1: Multiplicities in $\mathbb{C}[\Omega]_{\sqsupseteq 0}/\mathbb{C}[N(\overline{\Omega})]$ for the Fermat cubic, up to degree 8.

Note that b_λ can be computed with the software package [BKT12].

Denoting by a_λ the coefficients such that $\mathbb{C}[\Omega]_{\sqsupseteq 0} = \bigoplus_{\lambda\in\Lambda} \mathbb{V}(\lambda)^{a_\lambda}$, we are interested in the numbers $m_\lambda := a_\lambda - b_\lambda$, because:

$$\mathbb{C}[\Omega]_{\sqsupseteq 0}/\mathbb{C}[\overline{\Omega}] = \bigoplus\nolimits_{\lambda\in\Lambda} \mathbb{V}(\lambda)^{a_\lambda - b_\lambda}.$$

It follows from the Peter-Weyl Theorem (Theorem A.2.7) that $a_\lambda = \dim(\mathbb{V}(\lambda)^G)$, where $G \subseteq \mathrm{GL}_3$ is the stabilizer of P. It is well-known [BI15, Prop. 2.4] that G consists of permutation matrices and diagonal matrices whose diagonal entries are third roots of unity. A matrix representation of the canonical projection $\mathbb{V}(\lambda) \twoheadrightarrow \mathbb{V}(\lambda)^G$ with respect to the basis of semistandard Young tableaux (SSYT) is obtained by symmetrizing each SSYT with respect to G and straightening it [Ful97, § 7.4]. The quantity a_λ arises as the rank of this matrix. Using this method, we have computed the values of the $m_\lambda := a_\lambda - b_\lambda$ up to degree 8, see Figure 6.2.1 on page 70.

6.2.7 Remark. A formula for a_λ is more involved than the one for b_λ. Advancing methods used in [BI11, Section 4.2] (see also [Ike12, Section 5.2]), Ikenmeyer [Ike16] determined such a formula: For $\lambda \in \Lambda$, denote by $|\lambda| := \lambda_1 + \lambda_2 + \lambda_3$ the sum of its entries. We have $a_\lambda = 0$ unless $|\lambda| = 3d$ for some $d \in \mathbb{N}$. In this case,

$$a_\lambda = \sum_{\substack{\mu\in\Lambda \\ |\mu|=d}} \sum_{\substack{\nu_1,\dots,\nu_d\in\Lambda, \\ |\nu_k|=3\cdot k\cdot\hat{\mu}_k \\ \text{for all } k}} \mathrm{c}^{\lambda}_{\nu_1,\dots,\nu_d} \cdot \prod_{k=1}^{d} \mathrm{p}(\hat{\mu}_k, 3k; \nu_k), \tag{1}$$

where $\mathrm{c}^{\lambda}_{\nu_1,\dots,\nu_d}$ denotes the multi-Littlewood-Richardson coefficient and $\hat{\mu}_k$ denotes the number of times that k appears as an entry of μ. ●

6.3 Proof of Main Theorem

This section contains the proof of Theorem 6.2.3. We require results from Section A.3 and Subsection A.1.1.

Note that the stabilizer G acts on the variety $\mathrm{End}(W)$ by multiplication from the left. Since V is a polynomial $\mathrm{GL}(W)$-module, there is a well-defined morphism $\omega\colon \mathrm{End}(W) \to \overline{\Omega}$, $a \mapsto P \circ a$. For $h \in G$ and $a \in \mathrm{End}(W)$,

$$\omega(ha) = P \circ (ha) = P \circ h \circ a = P \circ a = \omega(a).$$

Therefore, ω is an G-invariant morphism. Since G is a reductive algebraic group, there is an affine quotient variety $\mathrm{End}(W)/\!\!/G$ together with a surjective morphism

$\pi\colon \operatorname{End}(W) \to \operatorname{End}(W)/\!\!/G$ and ω factors as a morphism $\phi\colon \operatorname{End}(W)/\!\!/G \to \overline{\Omega}$ such that the following diagram commutes:

$$\begin{tikzcd}
\operatorname{End}(W) \arrow[r, "\omega"] \arrow[d, two heads, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \overline{\Omega} \\
\operatorname{End}(W)/\!\!/G \arrow[ur, dotted, "\phi"'] &
\end{tikzcd} \tag{2}$$

Furthermore, the variety $\operatorname{End}(W)/\!\!/G$ is a normal variety because $\operatorname{End}(W)$ is normal [TY05, 27.5.1].

The morphism ϕ is a birational map. Indeed, $\operatorname{End}(W)/\!\!/G$ contains $\operatorname{GL}(W)/\!\!/G$ as an open subset [TY05, 27.5.2] and the restriction of ϕ to this open subset is the isomorphism $\operatorname{GL}(W)/\!\!/G \cong \Omega$, [TY05, 25.4.6]. Since Ω is an open subset of its closure [TY05, 21.4.3], this proves that ϕ is generically one to one.

The normalization $\nu\colon \mathrm{N}(\overline{\Omega}) \to \overline{\Omega}$ is a surjective, finite morphism of affine algebraic varieties [GW10, Proposition 12.43 and Corollary 12.52]. By the universal property of the normalization [GW10, Corollary 12.45] there exists a unique morphism $\psi\colon \operatorname{End}(W)/\!\!/G \to \mathrm{N}(\overline{\Omega})$ which completes (2) to a commutative diagram:

$$\begin{tikzcd}
\operatorname{End}(W) \arrow[r, "\omega"] \arrow[d, two heads, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \overline{\Omega} \\
\operatorname{End}(W)/\!\!/G \arrow[ur, "\phi"] \arrow[r, dotted, "\psi"'] \arrow[ur, phantom, shift right=3ex, "\circlearrowleft"] & \mathrm{N}(\overline{\Omega}) \arrow[u, two heads, "\nu"']
\end{tikzcd} \tag{3}$$

The morphism ψ is dominant and therefore corresponds to an injective ring homomorphism

$$\mathbb{C}[\mathrm{N}(\overline{\Omega})] \subseteq \mathbb{C}[\operatorname{End}(W)/\!\!/G] = \mathbb{C}[\operatorname{End}(W)]^G = (\mathbb{C}[\operatorname{GL}(W)]_{\sqsupseteq 0})^G,$$

due to Lemma A.3.4. Taking G-invariants is with respect to the left action of G on $\mathbb{C}[\operatorname{GL}(W)]$ and considering polynomial submodules is with respect to the right action of $\operatorname{GL}(W)$ on $\mathbb{C}[\operatorname{GL}(W)]$, so these two operations commute. Hence,

$$\mathbb{C}[\mathrm{N}(\overline{\Omega})] \subseteq (\mathbb{C}[\operatorname{GL}(W)]^G)_{\sqsupseteq 0} = \mathbb{C}[\Omega]_{\sqsupseteq 0}.$$

As a consequence, the polynomial part of the coordinate ring of Ω_P can be identified with the ring of G-invariants in $\mathbb{C}[\operatorname{End}(W)]$, where G is the stabilizer of P.

6.3.1 Remark. There is a commutative diagram of $\operatorname{GL}(W)$-equivariant inclusions of $\mathbb{C}$-algebras:

$$\begin{tikzcd}
\mathbb{C}[\operatorname{End}(W)] & \mathbb{C}[\overline{\Omega}] \arrow[l, hook'] \arrow[d, hook] \\
\mathbb{C}[\Omega]_{\sqsupseteq 0} = \mathbb{C}[\operatorname{End}(W)]^G \arrow[u, hook] & \mathbb{C}[\mathrm{N}(\overline{\Omega})] \arrow[l, hook']
\end{tikzcd} \tag{4}$$

Here, $\mathbb{C}[\overline{\Omega}]$ and $\mathbb{C}[\mathrm{End}(W)]^G$ have the same quotient field $\mathbb{K}$. The inclusion

$$\mathbb{C}[\mathrm{N}(\overline{\Omega})] \subseteq \mathbb{C}[\mathrm{End}(W)]^G = \mathbb{C}[\Omega]_{\sqsupseteq 0}$$

is an inclusion of integrally closed subrings of $\mathbb{K}$. (By definition, $\mathbb{C}[\mathrm{N}(\overline{\Omega})]$ is the integral closure of $\mathbb{C}[\overline{\Omega}]$ in $\mathbb{K}$.) ●

We now show that (1) of Theorem 6.2.3 implies $P \circ \mathrm{End}(W) = \overline{\Omega}$. Recall (3). The condition $\mathbb{C}[\mathrm{N}(\overline{\Omega})] = \mathbb{C}[\Omega]_{\sqsupseteq 0}$ holds if and only if the morphism ψ is an isomorphism. In this case it follows that ϕ is the normalization of $\overline{\Omega}$. Thus, (1) implies in particular that ϕ is surjective and therefore ω is surjective, which means $P \circ \mathrm{End}(W) = \overline{\Omega}$.

We now ask when the inclusion $\mathbb{C}[\mathrm{N}(\overline{\Omega})] \subseteq \mathbb{C}[\Omega]_{\sqsupseteq 0}$ becomes an equality. We will require an auxiliary lemma for the proof. The algebraic group $\mathbb{C}^\times = \mathrm{GL}_1$ acts polynomially on a variety X if the action morphism $\mathbb{C}^\times \times X \to X$ lifts to a morphism $\mathbb{C} \times X \to X$. We will denote this map by a dot, i.e., $(t, x) \mapsto t.x$.

6.3.2 Lemma. Let X and Y be affine varieties, each of them equipped with polynomial $\mathbb{C}^\times$-actions admitting unique fixed points $0_X \in X$ and $0_Y \in Y$, respectively. Let $\phi\colon X \to Y$ be a $\mathbb{C}^\times$-equivariant morphism. Then, $\phi^{-1}(0_Y) = \{0_X\}$ if and only if ϕ is finite.

Proof. The "only if" part is [Lan15, Lemma 7.6.3]. For the converse, assume that ϕ is finite. Let $x \in X$ be such that $\phi(x) = 0_Y$. Then, $\phi(t.x) = t.\phi(x) = t.0_Y = 0_Y$ for all $t \in \mathbb{C}^\times$ and hence, $\mathbb{C}^\times.x \subseteq \phi^{-1}(0_Y)$. But $\phi^{-1}(0_Y)$ is a finite set, therefore $\mathbb{C}^\times.x$ is finite and irreducible, i.e., a point. This implies $\mathbb{C}^\times.x = \{x\}$, so x is a fixpoint for the action of $\mathbb{C}^\times$. It follows that $x = 0_X$ by uniqueness of the fixpoint. ■

Lemma 6.3.2 will be applied to the morphism $\phi\colon \mathrm{End}(W)/\!\!/G \to \overline{\Omega}$. We therefore study the action of the scalar matrices $\mathbb{C}^\times \subseteq \mathrm{GL}(W)$ on $\mathrm{End}(W)/\!\!/G$ and $\overline{\Omega}$. Observe that the morphism ϕ is equivariant with respect to this action. We need to make sure that both varieties have a unique fixpoint in order to make use of Lemma 6.3.2.

We first reduce to the case where V has a unique fixpoint under the action of all scalar matrices. For this purpose, fix some basis of W, so $\mathrm{GL}(W) \cong \mathrm{GL}_n$ and let $V = \bigoplus_{\lambda \in \mathbb{N}^n} V_{(\lambda)}$ be the decomposition of V into isotypical components, i.e., $V_{(\lambda)}$ is a direct sum of irreducible modules of type λ. Note that the only weights λ that appear are in $\mathbb{N}^n$ because V is a polynomial GL_n-module. Let $P = \sum_{\lambda \in \mathbb{N}^n} P_\lambda$ be the corresponding decomposition of P, i.e., $P_\lambda \in V_{(\lambda)}$. Observe that the point $\tilde{P} := P - P_0$ has the same stabilizer as P, because any element of V_0 is $\mathrm{GL}(W)$-invariant. Let $\tilde{V} := \bigoplus_{\lambda \neq 0} V_{(\lambda)}$ be the complement of V_0 in V. Then, $\Omega \cong \{P_0\} \times \Omega_{\tilde{P}} \subseteq V_0 \times \tilde{V} \cong V$ and consequently, $\overline{\Omega}_P = \{P_0\} \times \overline{\Omega}_{\tilde{P}} \cong \overline{\Omega}_{\tilde{P}}$. This shows that we may henceforth assume $V = \tilde{V}$ and $P = \tilde{P}$. In this situation, the origin $0_V \in V$ is the only fixpoint under the action of the scalar matrices. Consequently, it is also the only $\mathbb{C}^\times$-fixpoint in $\overline{\Omega}$.

On the other hand, $\mathrm{End}(W)$ also has a unique fixpoint with respect to the left action by scalar matrices, namely the zero map which we will denote by 0. At this point, we require the following lemma to deduce that $\mathrm{End}(W)/\!\!/G$ also has a unique fixpoint:

6.3.3 Lemma. Let E be an affine variety on which $\mathbb{C}^\times$ acts polynomially with a unique fixpoint 0. Assume that a reductive group G acts on E from the left such that the actions of G and $\mathbb{C}^\times$ commute. Then, the quotient $E/\!\!/G$ also has a unique fixpoint under the induced action of $\mathbb{C}^\times$.

The proof of this lemma is slightly technical and will be given afterwards. Using it, we conclude that $\pi(0)$ is the unique fixpoint in $X := \mathrm{End}(W)/\!\!/G$ and 0_V is the unique fixpoint in $Y := \overline{\Omega}$. The morphism $\phi\colon X \to Y$ now satisfies the conditions of Lemma 6.3.2: We proceed to prove the equivalence of (1) and (2).

We first show (1) $\Rightarrow$ (2). If (1) holds, ψ is an isomorphism and ϕ is a normalization of $\overline{\Omega}$. Therefore, ϕ is a finite morphism. By one direction of Lemma 6.3.2, this implies that $\phi^{-1}(0_V) = \{\pi(0)\}$. In other words, $\phi(\pi(a)) = 0_V$ implies $\pi(a) = \pi(0)$. We have $P \circ a = \omega(a) = \phi(\pi(a))$, so $P \circ a = 0_V$ implies $\pi(a) = \pi(0)$, which is the same as saying that the zero map 0 is contained in the closure of the G-orbit of a. This is precisely (2).

For the converse implication, we assume (2). For any $a \in \mathrm{End}(W)$, the condition $0_V = \phi(\pi(a)) = \omega(a)$ implies $0 \in \overline{G \circ a}$ by (2). By construction of the GIT quotient, this implies $\pi(a) = \pi(0)$ and hence, $\phi^{-1}(0_V) = \{\pi(0)\}$. The other direction of Lemma 6.3.2 now states that ϕ is a finite morphism. Any finite morphism is integral [GW10, Remark 12.10], so ϕ is an integral birational map from a normal variety $\mathrm{End}(W)/\!\!/G$ to $\overline{\Omega}$. By [GW10, Proposition 12.44], it follows that it is the normalization of $\overline{\Omega}$, so ψ is an isomorphism.

Proof of Lemma 6.3.3. We first note that 0 is a fixpoint for the action of G as well. Indeed, for any $h \in G$ and any $t \in \mathbb{C}^\times$, we have $t.h.0 = h.t.0 = h.0$ because the actions commute, so $h.0$ is a fixpoint for the action of $\mathbb{C}^\times$. By uniqueness, this implies $h.0 = 0$. As $h \in G$ was arbitrary, 0 is a fixpoint for the action of G.

We will denote by $\pi\colon E \to E/\!\!/G$ the quotient morphism. Assume that $x = \pi(a)$ is any fixpoint of the action of $\mathbb{C}^\times$. Observe that

$$\{x\} = \mathbb{C}^\times.x = \mathbb{C}^\times.\pi(a) = \pi(\mathbb{C}^\times.a),$$

so $\mathbb{C}^\times.a \subseteq \pi^{-1}(x)$. Since the action of $\mathbb{C}^\times$ is polynomial, the orbit map lifts to a $\mathbb{C}^\times$-equivariant morphism $\gamma\colon \mathbb{C} \to E$ with $\gamma(t) = t.a$ for $t \in \mathbb{C}^\times$. Since

$$t.\gamma(0) = \gamma(t \cdot 0) = \gamma(0)$$

for all $t \in \mathbb{C}^\times$, it follows that $\gamma(0)$ is a $\mathbb{C}^\times$-fixpoint in E, so $\gamma(0) = 0$. This implies that $0 \in \overline{\mathbb{C}^\times.a}$. Because 0 is a fixpoint for the action of G, the set $\{0\} \subseteq E$ is a closed G-orbit. By the nature of the GIT-quotient, points of E that share a closed G-orbit are mapped to the same point in the quotient. In this case, a and 0 share the closed orbit $\{0\}$ and it follows that $x = \pi(a) = \pi(0)$. Thus, $\pi(0)$ is the only fixpoint of the action of $\mathbb{C}^\times$ on $E /\!\!/ G$. ■

Chapter 7

Techniques for Boundary Classification

We now turn to the more general case that $P \in \mathbb{C}[W]_d$ is not closed, so there exist forms in $\partial\Omega_P$ that are not of the form $P \circ a$ for any $a \in \mathrm{End}(W)$. Even so, any degeneration of P can be obtained by approximation, as we will explain in Section 7.1.

In several examples, some components of $\partial\Omega_P$ are orbit closures of degenerations of P. In some cases, like the 3×3 determinant, every component of the boundary contains a dense orbit – see Chapter 8. In fact, we have seen one such example in Lemma 6.2.5. If a degeneration Q of P is given, we can check whether $\overline{\Omega}_Q$ is a component of $\partial\Omega_P$ by testing whether $\dim(G_Q) = \dim(G_P) + 1$. The latter implies that $\overline{\Omega}_Q$ is a proper, G_P-invariant, codimension one subvariety of $\overline{\Omega}_P$, hence disjoint from Ω_P and consequently a component of $\partial\Omega_P$. In Section 7.2, we describe a way to compute the dimension of G_Q by means of linear algebra.

We can classify the components of $\partial\Omega_P$ in some cases by constructing all of them in this way. One way to ensure that all components have been found is to give a sharp upper bound on the number of components of $\partial\Omega_P$. The upper bound we will use is developped in Section 7.3.

7.1 Approximating Degenerations

Let $\mathbb{C}[t]$ be the polynomial ring over $\mathbb{C}$. For an element $q \in \mathrm{End}(W) \otimes \mathbb{C}[t]$, we write $q = \sum_{k=1}^{K} q_k t^k$ with $q_k \in \mathrm{End}(W)$, i.e., q_k is the k-th coefficient of q. On the other hand, one can also think of q as an endomorphism of the free $\mathbb{C}[t]$-module $W \otimes \mathbb{C}[t]$. Any form $P \in \mathbb{C}[W]$ can also be viewed as a polynomial with coefficients in $\mathbb{C}[t]$, giving meaning to the notation $P \circ q$.

7.1.1 Proposition. Let $P \in \mathbb{C}[W]_d$. There exists a natural number $K \in \mathbb{N}$ such that for every $Q \in \mathbb{C}[W]_d$, we have $Q \in \overline{\Omega}_P$ if and only if there exists a $q \in \mathrm{End}(W) \otimes \mathbb{C}[t]$ such that $P \circ q \equiv t^K \cdot Q \pmod{t^{K+1}}$.

Proof. We denote by $\mathbb{C}[\![t]\!]$ the ring of formal power series with complex coefficients. Furthermore, we denote by $\mathbb{C}(\!(t)\!)$ its fraction field, which consists of power series with finitely many terms of negative degree. Both rings contain $\mathbb{C}[t]$ in a natural way. By [LL89, Proposition 1 and the corollary on p. 11], there is a number $M \in \mathbb{N}$ with the following property: For every $Q \in \overline{\Omega}_P$, there exists $q \in \operatorname{End}(W) \otimes \mathbb{C}(\!(t)\!)$ such that we have $P \circ q \equiv Q \pmod{t}$ and $t^M q \in \mathbb{C}[\![t]\!]$. We define $K := dM$ and claim that it has the desired properties.

For the "$\Leftarrow$" direction, assume that $\omega(q) = P \circ q = t^K Q + \sum_{k>K} Q_k t^k$ with certain polynomials $Q_k \in \mathbb{C}[W]_d$. Let $(t_i)_{i\in\mathbb{N}}$ be a sequence of complex numbers that converges to zero. For every $i \in \mathbb{N}$, pick a d-th root ζ_i of t_i and define the endomorphism $a_i := \zeta_i^{-K} q(t_i) \in \operatorname{End}(W)$. Since P is homogeneous of degree d, we have $P \circ \zeta_i^{-K} \operatorname{id}_W = \zeta_i^{-dK} P = t_i^{-K} P$ and we may conclude that

$$P_i := P \circ a_i = t_i^{-K} \cdot (P \circ q(t_i)) = t_i^{-K} \left(t_i^K Q + \sum_{k>K} Q_k t_i^k \right) = Q + \underbrace{\textstyle\sum_{k\geq 1} Q_{k+K} t_i^k}_{\text{zero sequence}}.$$

Hence, $(P_i)_{i\in\mathbb{N}}$ is a sequence of elements in $\overline{\Omega}_P$ which converges to Q. Theorem 3.3.3 implies that $Q \in \overline{\Omega}_P$.

For the other direction, fix some $Q \in \mathbb{C}[W]_d$ and let $q = \sum_{k=-M}^{\infty} q_k t^k$ be such that $P \circ q \equiv Q \pmod{t}$. We then define $\bar{q} := \sum_{k=0}^{K} q_{k-M} t^k$ and note that it is sufficient to show

$$P \circ \bar{q} \equiv t^K \cdot Q \pmod{t^{K+1}}. \tag{1}$$

Recall that $V = \mathbb{C}[W]_d$ and let $E := \operatorname{End}(W)$. The orbit map $\omega \colon E \to V$ given by $\omega(q) = P \circ q$ is homogeneous of degree d, hence $\omega \in \mathbb{C}[E]_d \otimes V$. We can therefore write $\omega = \sum_{i=1}^{r} \omega_i \otimes Q_i$ with certain $\omega_i \in \mathbb{C}[E]_d$ and $Q_i \in V$.

Let $p \in \mathbb{C}[\![t]\!]$ be such that $t^M \cdot q = \bar{q} + t^{K+1} p$. Then,

$$\begin{aligned} \omega(q) &= \omega\left(t^{-M}\left(\bar{q} + t^{K+1}p\right)\right) = t^{-K} \cdot \omega\left(\bar{q} + t^{K+1}p\right) \\ &= \sum_{i=1}^{r} t^{-K} \cdot \omega_i\left(\bar{q} + t^{K+1}p\right) \cdot Q_i \\ &\equiv \sum_{i=1}^{r} t^{-K} \cdot \omega_i(\bar{q}) \cdot Q_i = t^{-K} \cdot \omega(\bar{q}) & \pmod{t}. \end{aligned}$$

Hence, $Q \equiv P \circ q = \omega(q) \equiv t^{-K} \cdot \omega(\bar{q}) = t^{-K} \cdot (P \circ \bar{q}) \pmod{t}$. Multiplying this equation by t^K yields (1). ∎

7.1.2 Remark. Note that for a particular Q, it can happen that the q from Proposition 7.1.1 is divisible by t, say $q = t^r p$. Then, we have

$$t^K Q \equiv P \circ q = P \circ (t^r p) = t^{dr} \cdot (P \circ p) \pmod{t^{K+1}}$$

so with $k := K - dr < K$, we have $t^k Q \equiv P \circ p \pmod{t^{k+1}}$. ●

7.1.3 Definition. Let $P, Q \in \mathbb{C}[W]_d$. A power series $q \in \operatorname{End}(W) \otimes \mathbb{C}[t]$ which satisfies $P \circ q \equiv t^k \cdot Q \pmod{t^{k+1}}$ is called an **approximation path** from P to Q and the number k is called its **order**. The minimum order of all approximation paths from P to Q is called the **approximation order** of Q with respect to P.

The **order of approximation** of $\overline{\Omega}_P$ is defined to be smalles number K that can be chosen in Proposition 7.1.1. It is also the maximum, taken over all $Q \in \overline{\Omega}_P$, of the approximation order of Q with respect to P. ●

7.1.4 Example. We give an example from the well-known study of ternary cubic forms, see [Kra85, I.7] for a complete classification. Compare also Lemma 6.2.5. We consider the homogeneous form $P = zx^2 - y^3 - z^3 \in \mathbb{C}[x, y, z]_3$. It defines an irreducible, nonsingular cubic curve and it is known that $\dim(G_P) = 0$. Let

$$q := \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix} \cdot t + \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix} \cdot t^3 = \operatorname{diag}(1, t, t^3) \in \operatorname{End}(\mathbb{C}^3) \otimes \mathbb{C}[t]\,.$$

Then, we have $k = 3$ and

$$P \circ q = (t^3 z)x^2 - (ty)^3 - (t^3 z)^3 = t^3 \cdot (zx^2 - y^3) - t^9 \cdot z^3$$

Note that $Q := zx^2 - y^3$ is the equation of a cusp, which is singular. In particular we have $Q \notin \Omega_P$ and one can compute $\dim(G_Q) = 1$ with the methods of Section 7.2. It follows that $\overline{\Omega}_Q$ is an irreducible component of $\partial\Omega_P$. ●

Since the set $\operatorname{End}(W) \otimes \mathbb{C}[t]$ is entirely too large to choose from, we need more information on how to choose approximation paths. There are only two straightforward observations we can make.

7.1.5 Remark. Let $P \in \mathbb{C}[W]_d$. An element $Q \in \overline{\Omega}_P$ has approximation order zero with respect to P if and only if $Q \in P \circ \operatorname{End}(W)$. In particular, P is closed if and only if $\overline{\Omega}_P$ has order of approximation zero. ●

7.1.6 Remark. Let $Q \in \overline{\Omega}_P$ have approximation order $k \geq 1$ and let $q = \sum_{i=0}^{K} q_i t^i$ be an approximation path from P to Q of order k. Then, $P \circ q_0 = 0$, because it is the coefficient of t in $P \circ q$. In other words, $q_0 \in \mathcal{A}_P$. ●

We also ask the following question:

7.1.7 Question. Is the order of approximation of $\overline{\Omega}_P$ related to the degree of the unique $\operatorname{SL}(W)$-invariant from Theorem 5.2.3?

If $P \in \mathbb{C}[W]_d$ and $Q \in \partial\Omega_P$, we say that Q is a **linear** degeneration of P if there is a **linear** approximation path from P to Q, i.e., one of the form $b + at$ with $b \in \mathcal{A}_P$. For many cases we consider, only linear degenerations occur.

7.2 The Lie Algebra Action

The **Lie algebra** of an algebraic group G is defined as the tangent space of G at the identity element $1 \in G$. We denote it by $\mathfrak{Lie}(G) := \mathrm{T}_1(G)$. It carries the structure of a Lie algebra, but this will not be of major importance for our considerations.

There is a strong relationship between the identity component G° of an algebraic group and its Lie algebra, c.f. [Hum98, 9] and [TY05, 23]. Moreover, there is the notion of a Lie algebra action on a vector space, and the action of an algebraic group induces an action of its Lie algebra as follows: The action of G on a space V is a morphism $\varrho\colon G \to \mathrm{GL}(V)$ of algebraic groups. Taking the derivative of this morphism at the identity yields a linear map $\mathrm{D}_\varrho\colon \mathfrak{Lie}(G) \to \mathfrak{Lie}(\mathrm{GL}(V)) \cong \mathrm{End}(V)$. The action of an element $a \in \mathfrak{Lie}(G)$ on $v \in V$ returns the vector $D_\varrho(a)(v) \in V$. We also use the common shorthands $\mathfrak{gl}(V) := \mathfrak{Lie}(\mathrm{GL}(V))$ and $\mathfrak{gl}_n := \mathfrak{Lie}(\mathrm{GL}_n)$.

By Proposition A.1.3 G is smooth, so we know that the dimension of $\mathfrak{Lie}(G)$ as a vector space is equal to the dimension of G as a variety. We will describe the vector space $\mathfrak{Lie}(G_P)$ explicitly in Corollary 7.2.3. Together with Theorem A.1.9, this also provides a way to compute the dimension of Ω_P.

Recall the partial derivative $\partial_w P$ of $P \in \mathbb{C}[W]_d$ in direction $w \in W$ from Definition 5.1.1.

7.2.1 Proposition. Let $V = \mathbb{C}[W]_d$ and let $\mathrm{GL}(W)$ act on V by precomposition. Then, the induced action of $\mathfrak{gl}(W) = \mathfrak{Lie}(\mathrm{GL}(W)) = \mathrm{End}(W) = W^* \otimes W$ on V is given by:

$$\begin{aligned} V \times \mathfrak{gl}(W) &\longrightarrow V \\ (P, a) &\longmapsto P * a \end{aligned}$$

where $P * (y \otimes w) = y \cdot \partial_w P$ for rank one tensors.

Proof. Let $\varrho\colon \mathrm{GL}(W) \to \mathrm{GL}(V)$ be the morphism of algebraic groups corresponding to the action of $\mathrm{GL}(W)$ on V. We have to compute the differential $\mathrm{D}_\varrho\colon \mathfrak{gl}(W) \to \mathfrak{gl}(V)$. Define $g(t) := \mathrm{id}_W + (ty \otimes w)$, it satisfies $g(0) = \mathrm{id}_W$ and $g'(0) = y \otimes w$. Consider a symmetric power $x^d \in \mathrm{Sym}^d W^* = V$ with $x \in W^*$ and observe

$$\varrho(g(t))(x^d) = x^d \circ g(t) = x^d \circ (\mathrm{id}_W + (ty \otimes w)) = (x + t \cdot x(w) \cdot y)^d.$$

The coefficient of t in this expression is $\mathrm{D}_\varrho(y \otimes w)(x^d)$. Expanding the right hand side, we can see that the coefficient of t is equal to $d \cdot y \cdot x(w) \cdot x^{d-1} = y \cdot \partial_w x^d$. Since symmetric powers span V, the result follows. ∎

7.2.2 Corollary. If we choose coordinates $\mathbb{C}[W] = \mathbb{C}[x_1, \ldots, x_n]$, this gives an action of GL_n on V. The corresponding action of $\mathfrak{gl}_n = \mathfrak{Lie}(\mathrm{GL}_n)$ in coordinates is given by

$$\begin{aligned} V \times \mathfrak{gl}_n &\longrightarrow V \\ (P, a) &\longmapsto \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \cdot x_j \cdot \partial_i P \end{aligned}$$

where $\partial_i P$ denotes the partial derivative of P with respect to x_i.

Proof. After choosing coordinates, a matrix $a = (a_{ij}) \in \mathbb{C}^{n \times n} \cong \mathfrak{gl}_n$ corresponds to the tensor $a = \sum_{i=1}^n \sum_{j=1}^n a_{ij} \cdot (x_j \otimes e_i)$, where $e_i \in W$ is the dual basis vector of $x_i \in W^*$. The result follows from Proposition 7.2.1. ∎

7.2.3 Corollary. Let $V = \mathbb{C}[x_1, \ldots, x_n]_d$ and let GL_n act on V by precomposition. For $P \in V$, we have

$$\mathfrak{Lie}(G_P) \cong \left\{ a \in \mathbb{C}^{n \times n} \;\middle|\; \sum_{j=1}^{n} \sum_{i=1}^{n} (a_{ij} \cdot x_j \cdot \partial_i P) = 0 \right\}$$

Proof. This follows from Corollary 7.2.2 and the fact that $\mathfrak{Lie}(G_P)$ is the set of all $a \in \mathfrak{gl}(W)$ that send P to zero, see [TY05, 25.1.3]. ∎

7.2.4 Example. Let $W := \mathbb{C}^{d \times d}$, so $\det_d \in \mathbb{C}[W]_d$. Let $H := G^\circ_{\det_d}$ be the identity component of the stabilizer of $\det_d$. We want to study $\mathfrak{Lie}(H)$. For a square matrix $A \in W$, we denote by $\mathrm{tr}(A) := \sum_{r=1}^d A_{rr}$ its **trace**.

There is a bilinear map $W \times W \to \mathrm{End}(W)$ defined by $(A, B) \mapsto A \otimes B$, where $A \otimes B$ denotes the map $C \mapsto ACB^{\mathrm{t}}$. It is straightforward to check that this induces an isomorphism $W \otimes W \cong \mathrm{End}(W)$. Let $E_{ij} \in W$ be the matrix that has the entry 1 in position (i, j) and zeros elsewhere. These matrices form a basis of W and the matrix representation of $A \otimes B$ with respect to this basis is the Kronecker product

$$(A \otimes B)_{(ij),(rs)} = (A \otimes B)(E_{rs})_{ij} = (A \cdot E_{rs} \cdot B^{\mathrm{t}})_{ij} = A_{ir} B^{\mathrm{t}}_{sj} = A_{ir} B_{js}.$$

Let $x_{ij} \in W^*$ be the dual vector of $E_{ij} \in W$, so that $\mathbb{C}[W]$ is a polynomial ring in the variables x_{ij}. Then $\det_d = \det(x)$ where $x = (x_{ij})$ is the matrix containing all the variables. We denote by $x^\sharp$ the **adjugate** of the matrix x, i.e., the transpose of the cofactor matrix of x. By definition, $x^\sharp_{sr} = \partial_{rs} \det_d$. By Corollary 7.2.2, the action of $A \otimes B \in \mathrm{End}(W) = \mathfrak{gl}(W)$ on the form $\det_d$ is given by

$$\sum_{i,j} \sum_{r,s} (A \otimes B)_{(ij),(rs)} \cdot x_{ij} \cdot x^\sharp_{sr} = \sum_{i,j,r,s} A^{\mathrm{t}}_{ri} x_{ij} B_{js} x^\sharp_{sr} = \mathrm{tr}(A^{\mathrm{t}} x B x^\sharp).$$

In summary, we have $A \otimes B \in \mathfrak{Lie}(H)$ if and only if $\operatorname{tr}(A^{\mathrm{t}} x B x^{\sharp}) = 0$ for all $x \in W$. Since $\mathrm{GL}_d \subseteq W$ is dense and $S^{\sharp} = \det(S) \cdot S^{-1}$ for all $S \in \mathrm{GL}_d$, we conclude:

$$A \otimes B \in \mathfrak{Lie}(H) \quad \Longleftrightarrow \quad \forall S \in \mathrm{GL}_d\colon\ \operatorname{tr}(A^{\mathrm{t}} S B S^{-1}) = 0. \tag{2}$$

Theorem 3.4.1 states that there is a surjective homomorphism of algebraic groups

$$\begin{aligned} \mathrm{SL}_d \times \mathrm{SL}_d &\twoheadrightarrow H, \\ (A, B) &\mapsto A \otimes B \end{aligned} \qquad \text{which induces} \qquad \begin{aligned} \mathfrak{sl}_d \times \mathfrak{sl}_d &\twoheadrightarrow \mathfrak{Lie}(H) \\ (A, B) &\mapsto (A \otimes \mathbb{I}) + (\mathbb{I} \otimes B). \end{aligned}$$

Note that $\mathfrak{sl}_d := \mathfrak{Lie}(\mathrm{SL}_d)$ is equal to the vector space of $d \times d$ matrices whose trace vanishes [TY05, 19.1]. Clearly, $(A \otimes \mathbb{I}) + (\mathbb{I} \otimes B)$ satisfies (2) when A and B are traceless, but it is not immediately obvious that any sum $\sum_{i=1}^{r} A_i \otimes B_i$ satisfying (2) must be of the form $(A \otimes \mathbb{I}) + (\mathbb{I} \otimes B)$ with A and B traceless. ●

7.3 Resolving the Rational Orbit Map

Recall the rational map ϖ_P from (1) in Section 5.3. We denote by $\operatorname{dom}(\varpi_P)$ its **domain**, i.e., the maximal open subset of $\mathbb{P}\operatorname{End}(W)$ on which ϖ is defined. There is a well-known classical way to resolve the indeterminacies of a rational map: One considers the graph

$$\Gamma := \Gamma(\varpi_P) = \overline{\left\{ \left(a, \varpi_P(a)\right) \mid a \in \operatorname{dom}(\varpi) \right\}} \subseteq \mathbb{P}\operatorname{End}(W) \times \mathbb{P}\overline{\Omega}_P$$

which has two natural morphisms $\beta\colon \Gamma \to \mathbb{P}\operatorname{End}(W)$ and $\gamma\colon \Gamma \to \mathbb{P}\overline{\Omega}_P$ induced by the projections. By definition, Γ is a projective variety. Since γ is a dominant projective morphism, it is surjective. With a good understanding of Γ, we could deduce a lot of information about $\mathbb{P}\overline{\Omega}_P$.

Unfortunately, while $\mathbb{P}\operatorname{End}(W)$ is just a projective space, the variety Γ and the morphism β are not well-understood in general. A priori, we only know that β is a blowup [Har06, II.7.17.3]. This is not very informative because any birational projective morphism is a blowup, see [Har06, II.7.17].

We want to give a minimal treatment here, even though one can define the blowup of any scheme in a closed subscheme, see [GW10, pp. 406] and [Har06, pp. 160]. For a classical treatment in the language of varieties, see [Har95, pp. 80]. We do require a hint of scheme language but want to avoid the technical overhead: Hence, we restrict to the affine case and heavily rely on the fact that blowups are local: See [GW10, Prop. 13.91(2)] and use that open immersions are flat [GW10, Prop. 14.3(4)].

7.3.1 Definition. Let X be an affine complex variety and $I \subseteq \mathbb{C}[X]$ an ideal. It corresponds uniquely to a subscheme $\hat{\mathcal{A}} := \operatorname{Spec}(\mathbb{C}[X]/I)$ of X. Choose generators $I = \langle \phi_0, \ldots, \phi_r \rangle$ and consider the rational map

$$\begin{aligned} \phi\colon X &\dashrightarrow \mathbb{P}^r \\ x &\longmapsto [\phi_0(x) : \ldots : \phi_r(x)] \end{aligned}$$

We denote by $\Gamma := \Gamma(\phi)$ the closure of $\{(x, \phi(x)) \mid x \in \operatorname{dom}(\phi)\} \subseteq X \times \mathbb{P}^r$, it is called the **graph** of ϕ. The projection $\beta\colon \Gamma \to X$ is called the **blowup** of X with **center** $\hat{\mathcal{A}}$. Up to an automorphism of Γ, this does not depend on the choice of the generators of I, see [Hü12, Prop. 1.19]. ●

Remark. We will usually denote by $\mathcal{A} \subseteq X$ the subvariety of X corresponding to the radical ideal $\sqrt{I}$, i.e., $\mathcal{A}$ is the support of $\hat{\mathcal{A}}$. ○

7.3.2 Definition. In line with the above definition, we denote by $\hat{\mathcal{A}}_P \subseteq \operatorname{End}(W)$ the closed subscheme of $\operatorname{End}(W)$ which corresponds to the ideal generated by the coefficients of $P \circ a$. Furthermore, we denote by $\hat{\mathcal{A}}_P^{\mathrm{ss}}$ the closed subscheme of the variety $\operatorname{End}(W)^{\mathrm{ss}}$ which is defined locally by the same equations as $\hat{\mathcal{A}}_P$. ●

Note that $\hat{\mathcal{A}}_P$ is supported on $\mathcal{A}_P = \{a \in \operatorname{End}(W) \mid P \circ a = 0\}$. The graph of ϖ_P is the projectivization of the blowup of $\operatorname{End}(W)$ with center $\hat{\mathcal{A}}_P$.

7.3.3 Example. In general, we do not have $\mathcal{A}_P = \hat{\mathcal{A}}_P$, i.e., the scheme $\hat{\mathcal{A}}_P$ is not reduced. Consider $W = \mathbb{C}^3$ and let $P \in \mathbb{C}[W]_3$ be any regular cubic. The affine cone $\mathrm{Z}(P) \subseteq W$ is an irreducible surface of degree 3, therefore it cannot contain any linear space of dimension two. It follows that the maximal linear subspaces of $\mathrm{Z}(P)$ are the lines contained in this cone and any nonzero $a \in \mathcal{A}_P$ is a rank one linear map whose image is spanned by a point $w \in \mathrm{Z}(P)$. Hence, under the canonical isomorphism $E := \operatorname{End}(W) = W \otimes W^*$, we have $a = w \otimes x$ for some $x \in W^*$. The projective class of a is a point in the segre embedding

$$\mathbb{P}^2 \times \mathbb{P}^2 \cong \mathbb{P}W \times \mathbb{P}W^* \subseteq \mathbb{P}(W \otimes W^*) = \mathbb{P}E$$

whose first coordinate is a point on the projective smooth curve defined by P, i.e., we have $\mathbb{P}\mathcal{A}_P = \mathbb{P}\,\mathrm{Z}(P) \times \mathbb{P}^2$. In particular, $\mathbb{P}\mathcal{A}_P$ is a smooth variety.

However, $\hat{\mathcal{A}}_P$ is not even a variety: On the one hand, the coefficients of $P \circ a$ are homogeneous polynomials of degree 3, so the ideal $I \subseteq \mathbb{C}[\operatorname{End}(W)]$ generated by them cannot contain any degree two polynomial. On the other hand, let $\mu \in \mathbb{C}[E]_2$ be any 2×2 minor, then μ vanishes on $\mathcal{A}_P = \mathrm{Z}(I)$ and therefore $\mu \in \sqrt{I}$. Since I contains no element of degree 2, we also have $\mu \notin I$. Hence, $I \neq \sqrt{I}$, so $\hat{\mathcal{A}}_P \neq \mathcal{A}_P$.

Note also that passing to semistable points does not change this fact: In this particular example, the stabilizer of P is a finite group by Lemma 6.2.5 and so every point is semistable. Thus, $\mathcal{A}_P^{\mathrm{ss}} \neq \hat{\mathcal{A}}_P^{\mathrm{ss}}$. ●

7.3.4 Remark. Example 7.3.3 generalizes as follows: Let $n := \dim(W)$ and $P \in \mathbb{C}[W]_d$. The hypersurface $Z(P)$ cannot contain a linear subspace of dimension $n-1$, so any endomorphism $a \in \mathcal{A}_P$ satisfies $\mathrm{rk}(a) < n-1$. Therefore, the ideal of $\hat{\mathcal{A}}_P$ contains some power of the $(n-1) \times (n-1)$ minors, which are polynomials of degree $n-1$. The coefficients of $P \circ a$ are homogeneous polynomials of degree d in the entries of a, so the ideal generated by them is not radical when $d \geq n$.

If P is generic, then $Z(P)$ is smooth. Assuming $d > 1$, it can be shown [Sha94, Exercise II.1.13] that every linear subspace of $Z(P)$ has dimension at most $\frac{n}{2}$, so $\hat{\mathcal{A}}_P$ is nonreduced already when $d \geq \frac{n}{2}$. ●

7.3.5. For the rest of this section, we set $E := \mathbb{P}\,\mathrm{End}(W)$ and $G := \mathbb{P}\,\mathrm{GL}(W)$ for brevity. A modest invariant of the boundary $\partial\Omega_P$ is its number of irreducible components. We are specifically interested in giving an upper bound on this number. We have the following commutative diagram:

$$\begin{tikzcd} E \arrow[rr, dashed, "\varpi_P"] & & \mathbb{P}\overline{\Omega}_P \\ & \Gamma \arrow[ul, "\beta"] \arrow[ur, two heads, "\gamma"'] & \end{tikzcd}$$

Define $U := \beta^{-1}(G) = \gamma^{-1}(\mathbb{P}\Omega_P)$ and let $Z := \Gamma \setminus U$ be its complement.

Claim. The number of irreducible components of Z is an upper bound on the number of irreducible components of $\partial\Omega_P$.

Proof of Claim. By commutativity, $U = \beta^{-1}(G) = \gamma^{-1}(\mathbb{P}\Omega_P)$ and the restriction of γ to U is an isomorphism onto $\mathbb{P}\Omega_P$. Since γ is also surjective, it follows that

$$\gamma(Z) = \gamma(\Gamma \setminus U) = \gamma(\Gamma \setminus \gamma^{-1}(\mathbb{P}\Omega_P)) = \mathbb{P}\overline{\Omega}_P \setminus \mathbb{P}\Omega_P = \mathbb{P}\partial\Omega_P. \tag{3}$$

Since $\mathbb{P}\partial\Omega_P$ has the same number of irreducible components as $\partial\Omega_P$, the statement follows. □

Unfortunately, this number does not seem easier to estimate than the number of components of $\partial\Omega_P$ itself. We will see later that passing to semistable points helps tremendously, which is what we will discuss next. ●

We give $V := \mathrm{End}(W) \otimes \mathbb{C}[W]_d$ a G_P-module structure by acting trivially on the right tensor factor. The graph Γ of ϖ_P admits the canonical embedding

$$\Gamma := \overline{\{([a], [P \circ a]) \mid a \notin \mathcal{A}_P\}} \subseteq \mathbb{P}\,\mathrm{End}(W) \times \mathbb{P}\overline{\Omega}_P \subseteq \mathbb{P}(V) \tag{4}$$

via the Segre map, see [Lan12, Def. 4.3.4.1] and the following discussion.

7.3.6 Lemma. The embedding (4) realizes Γ as a G_P-invariant subvariety of $\mathbb{P}(V)$. With the language of Subsection A.1.2, it turns Γ into a linearized G_P-variety.

Furthermore, Γ^{ss} is the graph of the restriction of ϖ_P to $\mathbb{P}\,\mathrm{End}(W)^{\mathrm{ss}}$.

Proof. We define $\Gamma' := \{([a],[P\circ a]) \mid a \notin \mathcal{A}_P\}$. It is an open, dense, G_P-invariant subset of Γ and therefore, $\Gamma = \overline{\Gamma'}$ is closed and G_P-invariant.

For the second claim, first note that a tuple $([a],[Q]) \in \Gamma$ is semistable if and only if $a \in \mathrm{End}(W)^{\mathrm{ss}}$ by definition of the group action. Hence, Γ^{ss} is the intersection of $\mathbb{P}\,\mathrm{End}(W)^{\mathrm{ss}} \times \mathbb{P}\overline{\Omega}_P$ with Γ. Since Γ^{ss} also contains Γ' by Proposition 5.3.4, an elementary topological argument yields the statement. ■

7.3.7 Definition. A blowup of a smooth variety with a smooth irreducible center of codimension at least two is called a **smooth** blowup. ●

Remark. The condition on the codimension is only to avoid pathologies. If the center has codimension one, the blowup is an isomorphism [Har06, II.7.14]. Note also that the center of a smooth blowup is always reduced, i.e., a variety [Eis94, Cor. 10.14]. ○

The following is our main tool for bounding the number of components of $\partial\Omega_P$. Restricting to semistable points will allow us to actually verify the assumption in some applications.

7.3.8 Proposition. Let $\beta\colon \Gamma^{\mathrm{ss}} \to (\mathbb{P}\,\mathrm{End}(W))^{\mathrm{ss}}$ be the projection from the graph of the restriction of ϖ_P to semistable points. If β factors as a sequence

$$\Gamma^{\mathrm{ss}} = Y_k \xrightarrow{\beta_k} \cdots \xrightarrow{\beta_2} Y_1 \xrightarrow{\beta_1} E^{\mathrm{ss}}$$

of k smooth blowups, then $\partial\Omega_P$ has at most $k+1$ irreducible components.

The proof of Proposition 7.3.8 will finish this section. We will require the following well-known result which can be found in [Har06, II.8.24]. Intuitively, it states that a smooth blowup only spawns a single irreducible component.

7.3.9 Proposition. Let $\beta\colon \Gamma \to X$ be a smooth blowup with center $\mathcal{A}$. Then, Γ is a smooth variety and $\beta^{-1}(\mathcal{A})$ is a smooth, irreducible, codimension one subvariety of Γ outside of which β is an isomorphism. ■

Note also that Proposition 7.3.8 would be easier to verify if we replaced E^{ss} by E, because the latter is projective. Since E^{ss} is usually *not* projective, neither is Γ^{ss} and we have to pass to a quotient to regain this property. We will discuss this process before proceeding to prove Proposition 7.3.8.

7.3.10. Clearly, $\beta\colon \Gamma \to E$ is G_P-equivariant and $\gamma\colon \Gamma \to \mathbb{P}\overline{\Omega}_P$ is G_P-invariant, in particular there is a unique morphism $\tilde{\gamma}\colon \Gamma^{\mathrm{ss}}/\!\!/G_P \to \mathbb{P}\overline{\Omega}_P$ such that

$$\begin{tikzcd} \Gamma^{\mathrm{ss}} \arrow[r, "\beta"] \arrow[d, two heads, "\pi"'] \arrow[dr, "\gamma"] & E^{\mathrm{ss}} \arrow[d, dashed, "\varpi_P"] \\ \Gamma^{\mathrm{ss}}/\!\!/G_P \arrow[r, "\tilde{\gamma}"'] & \mathbb{P}\overline{\Omega}_P \end{tikzcd} \tag{5}$$

commutes, by Proposition A.1.11. ●

7.3.11 Lemma. The morphism $\tilde{\gamma}$ in (5) is a birational, surjective morphism and its restriction to $\tilde{U} := \tilde{\gamma}^{-1}(\mathbb{P}\Omega_P)$ is an isomorphism $\tilde{U} \cong \mathbb{P}\Omega_P$.

Proof. Since ϖ_P and β are dominant, so is $\tilde{\gamma}$. The variety $\Gamma^{\mathrm{ss}} /\!\!/ G_P$ is projective, therefore $\tilde{\gamma}$ has closed image, so it is surjective. The orbit map ϖ_P restricts to a surjective morphism $\mathbb{P}\,\mathrm{GL}(W) \to \mathbb{P}\Omega_P$, and its graph is the open subset

$$U = \{([g], [P \circ g]) \mid g \in \mathrm{GL}(W)\} \subseteq \Gamma^{\mathrm{ss}}.$$

Hence, β restricts to am isomorphism $U \cong \mathbb{P}\,\mathrm{GL}(W)$ of G_P-varieties. We note here that $\tilde{U} = \pi(U)$ and $\tilde{\gamma}(\tilde{U}) = \gamma(U) = \mathbb{P}\Omega_P$, so $\tilde{\gamma}\colon \tilde{U} \to \mathbb{P}\Omega_P$ is a well-defined surjecive morphism. As $\mathbb{P}\Omega_P$ is smooth, $\tilde{\gamma}$ is an isomorphism if and only if it is injective, by Zariski's Main Theorem [TY05, 17.4.6]. Let $[Q] \in \mathbb{P}\Omega_P$, then we are left to show that the fiber $\tilde{\gamma}^{-1}([Q])$ contains exactly one element. To see this, let $g \in \mathrm{GL}(W)$ be such that $Q = P \circ g$. Then,

$$\tilde{\gamma}^{-1}([Q]) = \pi(\beta^{-1}(\varpi^{-1}([Q]))) = \pi(\beta^{-1}(G_P \circ [g])).$$

Since β is an equivariant isomorphism, $\beta^{-1}(G_P \circ [g])$ is an orbit. Since π is constant on orbits, we are done. ■

Proof of Proposition 7.3.8. We define $\tilde{\Gamma} := \Gamma^{\mathrm{ss}} /\!\!/ G_P$ and consider the diagram (5). We recall that $G = \mathbb{P}\,\mathrm{GL}(W) \subseteq \mathbb{P}\,\mathrm{End}(W) = E$ and $U = \gamma^{-1}(\mathbb{P}\Omega_P) = \beta^{-1}(G)$.

Set $Z := \Gamma^{\mathrm{ss}} \setminus U$. By Lemma 7.3.11, the morphism $\tilde{\gamma}\colon \tilde{\Gamma} \to \mathbb{P}\overline{\Omega}_P$ is birational, surjective, and restricts to an isomorphism on $\tilde{U} = \tilde{\gamma}^{-1}(\mathbb{P}\Omega_P) = \pi(U)$. We also define the set $\tilde{Z} := \pi(Z)$.

Claim. We have $\tilde{\gamma}(\tilde{Z}) = \mathbb{P}\partial\Omega_P$.

Proof of Claim. As every G_P-orbit in $U \cong G$ is closed, Property (G4) of the good quotient π implies that $\tilde{U}$ and $\tilde{Z}$ are disjoint. Since π is surjective, $\tilde{Z}$ is the complement of $\tilde{U}$ in $\tilde{\Gamma}$. The claim follows as in (3) because $\tilde{\gamma}$ is surjective by Lemma 7.3.11. □

As $\mathbb{P}\partial\Omega_P = \tilde{\gamma}(\tilde{Z}) = (\tilde{\gamma} \circ \pi)(Z)$ isthe image of Z under a morphism, we are left to verify that Z has at most $k+1$ irreducible components.

Recall that β is a composition

$$\Gamma^{\mathrm{ss}} = Y_k \xrightarrow{\beta_k} \cdots \xrightarrow{\beta_2} Y_1 \xrightarrow{\beta_1} Y_0 = E^{\mathrm{ss}}$$

of k smooth blowups. We set $\hat{\beta}_i := \beta_1 \circ \ldots \circ \beta_i$ for all $1 \leq i \leq k$. Furthermore, denote by $Z_0 := E^{\mathrm{ss}} \setminus G$ the set of projective classes of semistable endomorphisms that are noninvertible. Since E^{ss} is open in E, it follows that Z_0 is an irreducible hypersurface in E^{ss}. Let $Z_i := \hat{\beta}_i^{-1}(Z_0)$ for $1 \leq i \leq k$. Since the complement of $Z_k \subseteq Y_k = \Gamma^{\mathrm{ss}}$ is equal to $U = \beta^{-1}(G)$, we have $Z = Z_k$. We prove by induction on i that Z_i has at most

$i+1$ irreducible components, which is tautological for $i = 0$ and proves our initial claim for $i = k$. For the induction step, we assume $1 \leq i \leq k$ and that Z_{i-1} has at most i irreducible components.

Claim. We have $\mathcal{A}_i \subseteq Z_{i-1} \subseteq Y_{i-1}$, where $\mathcal{A}_i$ is the center of $\beta_i\colon Y_i \to Y_{i-1}$.

Proof of Claim. Assume for contradiction that $\mathcal{A}_i$ intersects $\hat{\beta}_{i-1}^{-1}(G)$. Since $\mathcal{A}_i$ is of codimension two and $\beta_i^{-1}(\mathcal{A}_i)$ is of codimension one by Proposition 7.3.9, the Fiber Dimension Theorem [TY05, 15.5.4] implies that there is a point $g \in G$ such that $\hat{\beta}_i^{-1}(g)$ has positive dimension. As all β_j are surjective, this would mean that $\beta^{-1}(g)$ has positive dimension, which is a contradiction. □

Since $\mathcal{A}_i \subseteq Z_{i-1}$, we get that

$$Z_i = \beta_i^{-1}(Z_{i-1}) = \beta_i^{-1}(Z_{i-1} \setminus \mathcal{A}_i) \cup \beta_i^{-1}(\mathcal{A}_i).$$

By Proposition 7.3.9, $\beta_i^{-1}(\mathcal{A}_i)$ is irreducible and $\beta_i^{-1}(Z_{i-1} \setminus \mathcal{A}_i) \cong Z_{i-1} \setminus \mathcal{A}_i$ has at most i irreducible components. Therefore, Z_i has at most $i+1$ irreducible components. ■

7.3.12 Remark. One can prove an even more refined statement than Proposition 7.3.8. Under the assumptions of the proposition, $Z := \Gamma^{\mathrm{ss}} \setminus \beta^{-1}(G)$ is a G_P-invariant closed subset of Γ^{ss}. The action of G_P on Γ^{ss} therefore permutes the finite set of irreducible components of Z. Denoting by ℓ the number of orbits of this action, it follows from the proof of Proposition 7.3.8 that the number of irreducible components of $\partial\Omega_P$ is at most $\ell + 1$. ●

Chapter 8

The 3 by 3 Determinant Polynomial

As outlined in Chapter 3, the orbit closure and boundary of the determinant polynomial is of particular interest for GCT. Yet, very few explicit results describing the geometry are known in low dimension. In this chapter we give a description of the boundary of the orbit of the 3×3 determinant. These results have been previously published in [HL16].

We view $\det_3$ as the polynomial

$$\det_3 := \det \left(\begin{smallmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{smallmatrix}\right) \in \mathbb{C}[x_1, \ldots, x_9]_3,$$

a homogeneous polynomial of degree 3 on the space $W := \mathbb{C}^{3\times 3}$. As before, we denote by $V := \mathbb{C}[W]_3$ the space of all homogeneous forms of degree 3 on W.

Our main result is a description of $\partial\Omega(\det_3)$ that answers a question of Landsberg [Lan15, Problem 5.4]: The two known components are the only ones. In Section 8.1 we explain the construction of the two components. Our contribution lies in Section 8.2 where we show that there is no other component.

8.0.1 Theorem. The boundary $\partial\Omega(\det_3)$ has exactly two irreducible components:

- The orbit closure of the determinant of the generic traceless matrix, namely

$$Q_1 := \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & -x_1 - x_5 \end{pmatrix};$$

- The orbit closure of the universal homogeneous polynomial of degree two in three variables, namely

$$Q_2 := x_4 \cdot x_1^2 + x_5 \cdot x_2^2 + x_6 \cdot x_3^2 + x_7 \cdot x_1 x_2 + x_8 \cdot x_2 x_3 + x_9 \cdot x_1 x_3.$$

Remark. The two components are different in nature: $\overline{\Omega}(Q_1) \subsetneq \det_3 \circ \operatorname{End}(W)$ is the orbit closure of a polynomial in only eight variables; the second component is more subtle and is not contained in $\det_3 \circ \operatorname{End}(W)$. This component has analogues in higher dimension and some results are known about them [LMR13]. ∘

8.1 Construction of Two Components of the Boundary

We denote by $\omega\colon \mathrm{End}(W) \to \overline{\Omega}(\det_3)$ the orbit map $\omega(a) = \det_3 \circ a$. Recall also the description of $G_{\det_3}$ from Theorem 3.4.1.

8.1.1 Lemma. We have $\dim(\Omega(\det_3)) = 65$ and $\dim(\Omega(Q_1)) = \dim(\Omega(Q_2)) = 64$.

Proof. The stabilizer $G_{\det_3}$ has dimension 16 by Theorem 3.4.1, thereby it follows that $\dim(\Omega_{\det_3}) = 81 - 16 = 65$ by Theorem A.1.9.

The dimension of $\Omega(Q_i)$ for $i \in \{1,2\}$ can be deduced from its Lie algebra. By Corollary 7.2.3, this amounts to computing the rank of a 165×81 matrix, which is easy using a computer, see Program 8.1. ∎

```
from sympy import *

def OrbitDimension(P,X):
    P = Poly(P,X)
    Df = [ Poly(x)*diff(P,y) for x in X for y in X ]
    Mn = list(set(sum((Poly(Q).monoms() for Q in Df),[])))
    return Matrix([[ Q.coeff_monomial(m) for Q in Df ]
        for m in Mn ]).rank()

x,y,z,a,b,c,d,e,f = X = symbols('x y z a b c d e f')

Q1 = Matrix([[x,y,z],[a,b,c],[d,e,-x-b]]).det()
Q2 = a*x**2 + b*y**2 + c*z**2 + d*x*y + e*y*z + f*z*x

print(OrbitDimension(Q1,X))
print(OrbitDimension(Q2,X))
```

Program 8.1: Stabilizer computation with *Python* [Pyt; Sym].

8.1.2 Lemma. $\overline{\Omega}(Q_1)$ is an irreducible component of $\partial\Omega(\det_3)$.

Proof. Since $\det_3$ is concise and Q_1 is not, we have $\overline{\Omega}(Q_1) \subseteq \partial\Omega(\det_3)$. Lemma 8.1.1 implies that $\overline{\Omega}(Q_1)$ has codimension one in $\overline{\Omega}(\det_3)$ and must therefore be an irreducible component of $\partial\Omega(\det_3)$. ∎

8.1.3 Lemma. $\overline{\Omega}(Q_2)$ is an irreducible component of $\partial\Omega(\det_3)$, distinct from $\overline{\Omega}(Q_1)$.

Proof. It is easy to see that Q_2 is concise whereas Q_1 is not, so $\overline{\Omega}(Q_1)$ contains no concise polynomial, but $\overline{\Omega}(Q_2)$ does. It follows that the two orbit closures are distinct. Let

$$b := \begin{pmatrix} 0 & x_1 & -x_2 \\ -x_1 & 0 & x_3 \\ x_2 & -x_3 & 0 \end{pmatrix} \qquad \text{and} \qquad a := \begin{pmatrix} 2x_6 & x_8 & x_9 \\ x_8 & 2x_5 & x_7 \\ x_9 & x_7 & 2x_4 \end{pmatrix}.$$

Th entries of these matrices are linear forms on W, so $a, b \in \operatorname{End}(W)$ with b projecting onto the space of antisymmetric matrices and a projecting onto its orthogonal complement of symmetric matrices.

Recall Section 7.1. To show that $Q_2 \in \partial\Omega(\det_3)$, we will use the approximation path $b + at$ as outlined. It is clear that $b \in \mathcal{A}_{\det_3}$. The coefficient of t in $\det(b + ta)$ is equal to $\operatorname{tr}(b^\sharp a)$ by Jacobi's formula, where $b^\sharp$ is the adjugate matrix of b. Furthermore, we have $b^\sharp = u^t u$ with $u = (x_3, x_2, x_1)$. Since $\operatorname{tr}(b^\sharp a) = uau^t = 2Q_2$, we have $[Q_2] \in \mathbb{P}\overline{\Omega}(\det_3)$ by Proposition 7.1.1. Lemma 8.1.1 implies that $\overline{\Omega}(Q_2)$ has the same dimension as $\partial\Omega(\det_3)$, so it is one of its irreducible components. ■

Note that Lemma 8.1.3 generalizes to higher dimensions: the limit of the determinant on the space of skew-symmetric matrices always leads to a component of the boundary of the orbit of $\det_d$, when $d \geq 3$ is odd, as shown by Landsberg, Manivel, and Ressayre [LMR13, Prop. 3.5.1].

8.2 There Are Only Two Components

Throughout this section, we will denote by $G := G_{\det_3}$ the stabilizer group of $\det_3$.

Recall the contents of Section 5.3. The annihilator $\mathcal{A} := \mathcal{A}_{\det_3}$ is precisely known, thanks to the classification of the maximal linear subspaces of W containing only singular matrices [Atk83; FLR85; EH88]. These spaces are precisely the maximal linear subspaces of $Z(\det_3)$, see also Remark 5.3.2.

For every $a \in \mathcal{A}$, there is a $h \in G^\circ$ such that $\operatorname{im}(h \circ a)$ is a subset of one of the following spaces of singular matrices:

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \alpha & -\beta \\ -\alpha & 0 & \gamma \\ \beta & -\gamma & 0 \end{pmatrix} \text{ for } \alpha, \beta, \gamma \in \mathbb{C}.$$

The first three are called compression spaces, we denote them by L_1, L_2 and L_3. The fourth is the space of 3×3 skew-symmetric matrices, which we will denote by L_0. Let us write

$$\operatorname{End}(W, L) := \{a \in \operatorname{End}(W) \mid \operatorname{im}(a) \subseteq L\} \tag{1}$$

for a linear subspace $L \subseteq W$. The sets $E_i := G^\circ \circ \operatorname{End}(W, L_i)$ constitute the four irreducible components of $\mathcal{A}$. For example,

$$E_0 = \{[a] \in \mathbb{P}(E) \mid \exists S, T \in \mathrm{SL}_3\colon \ \operatorname{im}(a) \subseteq S \cdot L_0 \cdot T\}.$$

Indeed, they are irreducible because they are the image of the irreducible variety $G^\circ \times \operatorname{End}(W, L_i)$ under the action morphism. Also, the E_i are not contained in one another and $\mathcal{A}$ is their union.

We denote by E_i^{ss} the set of semistable endomorphisms in E_i.

8.2.1 Lemma. We have $\mathcal{A}^{\mathrm{ss}} = E_0^{\mathrm{ss}} \neq \emptyset$.

Proof. Let $g_t := \operatorname{diag}(t, t, t^{-2})$, then $g_t \in \mathrm{SL}_3$ for all $t \in \mathbb{C}$ and

$$g_t \cdot \left(\begin{smallmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{smallmatrix}\right), \qquad \left(\begin{smallmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{smallmatrix}\right) \cdot g_t \qquad \text{and} \qquad g_t^{-1} \cdot \left(\begin{smallmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{smallmatrix}\right) \cdot g_t^{-1}$$

all tend to 0 when $t \to 0$, for any constants $*$. This proves that the first three components do not meet $\operatorname{End}(W)^{\mathrm{ss}}$. To show that $\mathcal{A}^{\mathrm{ss}}$ is not empty, we exhibit an invariant function $f \in \mathbb{C}[\operatorname{End}(W)]^G$, which does not vanish at a point $b \in \operatorname{End}(W, L_0)$. Pick any three matrices $A, B, C \in W$ and let $\tilde{f}\colon \operatorname{End}(W) \to \mathbb{C}$ be the regular function

$$\tilde{f}(a) := \operatorname{tr}\Big(a(A) \cdot a(B)^\sharp \cdot a(C) \cdot a(A+B+C)^\sharp\Big).$$

The dot here is multiplication of 3×3 matrices. This is a G°-invariant polynomial of degree 6. Indeed, if $h \in G^\circ$ is the map $A \mapsto SAT$ for some $S, T \in \mathrm{SL}_3$, then

$$\tilde{f}(h \circ a) = \operatorname{tr}\Big(S \cdot a(A) \cdot T \cdot T^\sharp \cdot a(B)^\sharp \cdot S^\sharp \cdot S \cdot a(C) \cdot T \cdot T^\sharp \cdot a(A+B+C)^\sharp \cdot S^\sharp\Big) = \tilde{f}(a)$$

because $S^\sharp = S^{-1}$ and $T^\sharp = T^{-1}$. It follows that $f := \tilde{f} + (\tilde{f} \circ \tau)$ is G-invariant, where $\tau \in \operatorname{End}(W)$ is the transposition map $\tau(A) := A^{\mathrm{t}}$. We now provide a point $b \in \operatorname{End}(W)$ with $f(b) \neq 0$. Let b be the projection $b\colon W \twoheadrightarrow L_0$ which has the following description in coordinates:

$$b = \begin{pmatrix} 0 & x_1 & -x_2 \\ -x_1 & 0 & x_3 \\ x_2 & -x_3 & 0 \end{pmatrix}. \tag{2}$$

For generic choices $A, B, C \in W$, a simple computation shows that $f(b) \neq 0$.

```
from sympy import Matrix

A = Matrix([[1,1,1],[1,1,1],[1,1,1]])
B = Matrix([[1,0,1],[0,1,0],[1,0,1]])
C = Matrix([[0,0,1],[0,1,0],[1,0,0]])

b = lambda A: (lambda x: Matrix(
       [[    0 ,  x[0], -x[1] ],
        [ -x[0],    0 ,  x[2] ],
        [  x[1], -x[2],    0  ]]) )(list(A))

t = lambda a: ( a(A) * (a(B).adjugate()) * a(C) *
  (a(A+B+C).adjugate()) ).trace()
f = lambda a: t(a) + t(lambda A: a(A).transpose())
print(f(b))
```

The above *Python* program [Pyt; Sym] evaluates f on b for a particular choice of matrices A, B and C giving $f(b) = 2$. ■

8.2.2 Lemma. An endomorphism $a \in \mathcal{A}^{\mathrm{ss}}$ satisfies $\operatorname{rk}(a) \geq 3$.

Proof. In [BD06, Thm. 2 and discussion above it], Bürgin and Draisma show that any 2-dimensional subspace of E containing only singular matrices is contained in a compression space. Therefore, if the image of a had dimension 2 or less, then a would lie in the nullcone, which is contrary to the choice of a. ■

Recall Section 7.3, in particular Definition 7.3.1. We set $\hat{\mathcal{A}}^{\mathrm{ss}} := \hat{\mathcal{A}}^{\mathrm{ss}}_{\det_3}$, so $\mathbb{P}\hat{\mathcal{A}}^{\mathrm{ss}}$ are the indeterminacies of $\varpi\colon \mathbb{P}\operatorname{End}(W) \to \mathbb{P}\overline{\Omega}(\det_3)$. The following proposition in conjuction with Proposition 7.3.8 implies that $\partial\Omega(\det_3)$ has at most two irreducible components, both of which we have described in Section 8.1. This finishes the proof of Theorem 8.0.1.

8.2.3 Proposition. The subscheme $\mathbb{P}\hat{\mathcal{A}}^{\mathrm{ss}} \subseteq \mathbb{P}\operatorname{End}(W)$ is a smooth subvariety. In particular, the projection $\beta\colon \Gamma^{\mathrm{ss}} \to \mathbb{P}\operatorname{End}(W)^{\mathrm{ss}}$ is a smooth blowup, where Γ^{ss} denotes the graph of ϖ restricted to $\mathbb{P}\operatorname{End}(W)^{\mathrm{ss}}$.

Proof. We set $E := \operatorname{End}(W)$ to shorten notation. By Definition 7.3.1 and Lemma 7.3.6, Γ^{ss} is the blowup of $\mathbb{P}E^{\mathrm{ss}}$ along $\mathbb{P}\hat{\mathcal{A}}^{\mathrm{ss}}$. Therefore, we have to check that $\mathbb{P}\hat{\mathcal{A}}^{\mathrm{ss}}$ is irreducible and smooth, the latter also implies that it is reduced, i.e., $\hat{\mathcal{A}}^{\mathrm{ss}} = \mathcal{A}^{\mathrm{ss}}$.

Since scalar matrices stabilize any point in $\mathbb{P}(V)$, the stabilizer of $[\det_3] \in \mathbb{P}(V)$ is the group $H := G \circ \mathbb{C}\,\mathrm{id}_W \subseteq \mathrm{GL}(W)$.

Claim. Let $b \in \mathcal{A}^{\mathrm{ss}}$ be the point defined in (2). We claim $\mathcal{A}^{\mathrm{ss}} = H^\circ \circ b \circ \mathrm{GL}(W)$, the orbit of b under the action of $H^\circ \times \mathrm{GL}(W)$ by multiplication from left and right.

Proof of Claim. For "$\supseteq$", note that the left hand side is invariant under both actions and contains b. Conversely, let $a \in \mathcal{A}^{\mathrm{ss}}$. By Lemma 8.2.1 and because L_0 is invariant under transposition, we may assume that the image of a is included in L_0, up to replacing a by another point in its orbit $H^\circ \circ a$. By Lemma 8.2.2, we also know that $\operatorname{rk}(a) \geq 3$, so a surjects onto L_0. This implies that there is some $g \in \mathrm{GL}(W)$ such that $a = b \circ g$, and thus $a \in H^\circ \circ b \circ \mathrm{GL}(W)$. □

In particular, $\mathcal{A}^{\mathrm{ss}}$ is irreducible, and smooth by Theorem A.1.9.(1). We will now show that $\hat{\mathcal{A}}^{\mathrm{ss}}$ is also smooth. Since $\hat{\mathcal{A}}^{\mathrm{ss}}$ is invariant under the actions of both H and $\mathrm{GL}(W)$ and its support $\mathcal{A}^{\mathrm{ss}}$ is an orbit under the same action, it suffices to verify that $\hat{\mathcal{A}}^{\mathrm{ss}}$ is smooth at one *single* point, say b. This amounts to checking that the dimension of the tangent space $\mathrm{T}_b\,\hat{\mathcal{A}}^{\mathrm{ss}}$ equals the dimension of $\mathcal{A}^{\mathrm{ss}}$.

Recall that $\hat{\mathcal{A}}$ is the scheme given by the coefficients of $\det_3 \circ a = 0$, as polynomials in the entries of a. By the Jacobian criterion [EH00, §V.3], we obtain the description

$$\mathrm{T}_b\,\hat{\mathcal{A}}^{\mathrm{ss}} \cong \left\{ c \in E \;\middle|\; t^2 \text{ divides } \det_3 \circ (b + tc) \right\},$$

where t is a formal variable. The dimension of $\mathrm{T}_b\,\hat{\mathcal{A}}^{\mathrm{ss}}$ can be determined by means of Program 8.2: It is equal to 35.

To calculate the dimension of $\mathcal{A}^{\mathrm{ss}}$, we use the fact that it is an orbit under the action of $H^\circ \times \mathrm{GL}(W)$. More precisely, we consider the derivative of the orbit map

$$\begin{aligned} H^\circ \times \mathrm{GL}(W) &\longrightarrow \mathcal{A}^{\mathrm{ss}} \\ (a, c) &\longmapsto a \circ b \circ c \end{aligned}$$

at the neutral element, yielding a surjective linear map

$$\begin{aligned} \mathfrak{Lie}(H^\circ) \times \mathfrak{gl}(W) &\longrightarrow \mathrm{T}_b\,\mathcal{A}^{\mathrm{ss}} \\ (a, c) &\longmapsto ab + bc \end{aligned}$$

We have $\mathfrak{gl}(W) \cong E$. Recall the Lie algebra of G from Example 7.2.4. Similar to that example, we can check that the Lie algebra of H° is given by the maps $(S \otimes \mathbb{I}) + (\mathbb{I} \otimes T)$ for $S, T \in W$ – i.e., there is no condition on S and T to be traceless because H° contains scalar matrices. We can therefore express the tangent space as

$$\begin{aligned} \mathrm{T}_b\,\mathcal{A}^{\mathrm{ss}} &= \{ab + bc \mid a \in \mathfrak{Lie}(H),\ c \in \mathfrak{gl}(W)\} \subset \mathrm{T}_b\,E \cong E \\ &= \{(S \otimes \mathbb{I}) \circ b + (\mathbb{I} \otimes T) \circ b + b \circ c \mid S, T \in W,\ c \in E\}\,. \end{aligned}$$

In other words, $\mathrm{T}_b\,\mathcal{A}^{\mathrm{ss}}$ consists of all maps of the form

$$\begin{aligned} W &\longrightarrow W \\ C &\longmapsto S \cdot b(C) + b(C) \cdot T + b(c(C)) \end{aligned}$$

for certain $c \in E$ and $S, T \in W$. Program 8.2 also verifies that this space has dimension 35, which terminates the proof. ■

```
from sympy import *
from sympy.abc import t

b = lambda A: (lambda x: Matrix(
    [[    0 ,  x[0], -x[1] ],
     [ -x[0],    0 ,  x[2] ],
     [  x[1], -x[2],    0  ]]) )(list(A))

generic_matrix = lambda v,n: ( lambda s:
    Matrix([ s[i::n] for i in range(n) ]) )(
     symbols('%s:%d'%(v,n*n)) )

A = generic_matrix('a',3)
c = generic_matrix('c',9)

cA = list(c*Matrix(list(A)))
cA = Matrix( [cA[i::3] for i in range(3)] )
cV = list(c)

D = (b(A) + t * cA).det()
L = Poly( diff(D,t).subs(t,0), list(A) ).coeffs()
L = [ collect(l,cV) for l in L ]
M = Matrix([[ term.coeff(i) for i in cV ] for term in L ])

# dim(End(W)) = 81
print("tangent space of indeterminacy:", 81-M.rank())

S = generic_matrix('s',3)
T = generic_matrix('t',3)
V = list(S) + list(T) + cV
B =  S*b(A) + b(A)*T + b(cA)
L = [t for l in list(B) for t in Poly(l,list(A)).coeffs()]
N = Matrix([[ term.coeff(i)  for i in V ] for term in L ])

print("tangent space of HxGL(W)-orbit:", N.rank())
```

Program 8.2: Tangent space computation with *Python* [Pyt; Sym].

8.3 The Traceless Determinant

A computation as in Lemma 8.1.2 shows that for $d \in \{3,4,5\}$, the orbit closure of the traceless $d \times d$ determinant is a component of $\partial\Omega(\det_d)$. We will show in this section that it is true for all $d \geq 3$. The main work in proving this claim is the following description of the stabilizer of the traceless determinant. In fact, its dimension is the only information required, but a complete description is certainly of independent interest.

8.3.1 Theorem. Let $W = \left\{ A \in \mathbb{C}^{d\times d} \,\middle|\, \mathrm{tr}(A) = 0 \right\}$ with $d \geq 3$ and let $P \in \mathbb{C}[W]_d$ be the restriction of $\det_d$ to W. The stabilizer group G_P is reductive of dimension $d^2 - 1$. Moreover,

(1) The identity component of the stabilizer of P is the group

$$G_P^\circ = \left\{ g \in \mathrm{GL}(W) \,\middle|\, \exists S \in \mathrm{SL}_d \colon \forall A \in W \colon g(A) = SAS^{-1} \right\}.$$

(2) Let $\Delta \subseteq \mathrm{GL}(W)$ be the group generated by

- the transposition map $t\colon W \to W$, $t(A) = A^{\mathrm{t}}$
- and all $\zeta \cdot \mathrm{id}_W$ where $\zeta \in \mathbb{C}$ is a d-th root of unity.

Then, $G_P = \Delta \cdot G_P^\circ$ and G_P has $|\Delta| = 2d$ connected components.

The proof is postponed briefly in order to first draw the announced conclusion.

8.3.2 Corollary. The orbit closure of the traceless determinant is an irreducible component of $\partial\Omega(\det_d)$, for all $d \geq 3$.

For the proof of this corollary, we also require the following technical lemma, which we will prove in direct succession.

8.3.3 Lemma. Let $M \cong \mathbb{C}^n$, $D \in \mathbb{C}[M]_d$ a concise form and $W \subseteq M$ a linear subspace of codimension k. Let $a \in \mathrm{End}(M, W)$ be any projection onto W, let $Q := D \circ a$ and denote by $P \in \mathbb{C}[W]_d$ the restriction of Q to W. Then, $\dim(G_Q) = \dim(G_P) + kn$.

Proof of Corollary 8.3.2. We set $M := \mathbb{C}^{d\times d}$, $D := \det_d \in \mathbb{C}[M]_d$ and $W := \ker(\mathrm{tr})$. We use the notation in Lemma 8.3.3, where $k = 1$ and $n = d^2$. By Theorem 8.3.1 we have $\dim(G_P) = n - 1$ and Lemma 8.3.3 states that $\dim(G_Q) = n - 1 + n = 2n - 1$. By Theorems A.1.9 and 3.4.1 this implies $\dim(\Omega_Q) = \dim(\Omega_{\det_d}) - 1$. ∎

Proof of Lemma 8.3.3. We can choose coordinates $\mathbb{C}[M] = \mathbb{C}[x_1, \ldots, x_n]$ such that the x_i with $i \leq n - k$ are coordinates on W. Then, Q is simply the polynomial that arises from D by setting x_i to zero for $i > n - k$, in the chosen coordinates. By Corollary 7.2.3,

$$\mathfrak{Lie}(G_Q) = \left\{ a \in \mathbb{C}^{n\times n} \,\middle|\, \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_j\partial_i Q = 0 \right\} = \left\{ a \in \mathbb{C}^{n\times n} \,\middle|\, \sum_{i=1}^{n-k}\sum_{j=1}^{n} a_{ij}x_j\partial_i Q = 0 \right\}$$

because $\partial_i Q = 0$ for $i > n-k$. It suffices to show that $\dim(\mathfrak{Lie}(G_Q)) = \dim(G_P) + kn$. We have

$$\begin{aligned}
&\dim\left(\left\{ a \in \mathbb{C}^{n\times n} \,\middle|\, \sum_{i=1}^{n-k}\sum_{j=1}^{n-k} a_{ij}x_j\partial_i Q = 0 \right\}\right) \qquad (3)\\
&= \dim\left(\left\{ a \in \mathbb{C}^{(n-k)\times(n-k)} \,\middle|\, \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} a_{ij}x_j\partial_i P = 0 \right\}\right) + (n^2 - (n-k)^2)\\
&= \dim(\mathfrak{Lie}(G_P)) + 2kn - k^2
\end{aligned}$$

We will show that $\langle x_j\partial_i Q \mid 1 \le i \le n-k < j \le n, \rangle_{\mathbb{C}}$ has dimension $k(n-k)$ and trivial intersection with $\langle x_j\partial_i Q \mid 1 \le i,j \le n-k\rangle_{\mathbb{C}}$. This implies that $\mathfrak{Lie}(G_Q)$ arises from (3) by imposing $k(n-k) = kn - k^2$ additional linearly independent conditions, dropping the dimension from $\dim(G_P) + 2kn - k^2$ to $\dim(G_P) + kn$ as claimed.

The spaces intersect trivially because all monomials of polynomials in the second space are divisibly by some x_j with $j \le n-k$ and none of the monomials of polynomials in the first space are.

As a is the identity on W, we have $\partial_i Q = \partial_i D$ for $i \le n-k$ and the $\partial_i D$ are linearly independent by Proposition 5.1.2 because D is concise. It follows that the $x_j\partial_i Q$ with $1 \le i \le n-k$ and $n-k < j \le n$ are all linearly independent. ∎

8.3.1 Proof of Theorem 8.3.1

The thesis of Reichenbach [Rei16] contains a very detailed proof of Theorem 3.4.1, the description of the stabilizer of $\det_d$. It can be adapted to verify Theorem 8.3.1, and some parts of the proof require only slight modification or none at all. We therefore refer to Reichenbach [Rei16] frequently and also make use of partial results therein.

We will denote by $M := \mathbb{C}^{d\times d}$ the space of square matrices and recall that $W \subseteq M$ is the subspace of traceless matrices. We will denote by $E_{ij} \in M$ the matrix with entry 1 in position (i,j) and zeros elsewhere. We begin with the important observation that we only have to show $G_P \subseteq G_{\det_d}$:

8.3.4 Lemma. Let $g \in G_P$. If there are $S, T \in \mathrm{GL}_d$ such that $g(A) = S \cdot A \cdot T$ for all $A \in W$, then there is a d-th root of unity $\zeta \in \mathbb{C}$ such that $T = \zeta \cdot S^{-1}$.

Proof. Set $I := TS$. Since $g \in \mathrm{GL}(W)$, we have $SAT \in W$ for any $A \in W$. Hence,

$$\forall A \in W\colon\ 0 = \mathrm{tr}(SAT) = \mathrm{tr}(ATS) = \mathrm{tr}(AI).$$

Let E_{ij} be the $d \times d$ matrix with entry 1 in position (i,j) and zeros elsewhere. For distinct indices $i \neq j$, we have $0 = \mathrm{tr}(E_{ij}I) = I_{ij}$ and therefore I is a diagonal matrix.

Furthermore, from $0 = \operatorname{tr}((E_{ii} - E_{jj})I) = I_{ii} - I_{jj}$ it follows that all diagonal entries of I are equal to some $\zeta := I_{11}$. Finally, pick some $A \in W$ with $\det(A) \neq 0$. From

$$\det(A) = \det(g(A)) = \det(SAT) = \det(S) \cdot \det(A) \cdot \det(T) = \det(TS) \cdot \det(A),$$

it follows that $1 = \det(TS) = \det(I) = \zeta^d$. ∎

Recall that a singular subspace of M is a linear space $L \subseteq M$ such that $\det(A) = 0$ for all $A \in L$. We denote by $\mathcal{L}$ the set of all maximal left ideals of M as a ring. By [Rei16, Satz 2.24], there is a bijection $\mathbb{P}^{d-1} \cong \mathcal{L}$ given by $v \mapsto L(v)$, where

$$L(v) := \{A \in M \mid v \subseteq \ker(A)\}.$$

Here, v denotes a line in $\mathbb{C}^d$. In particular, every element of $\mathcal{L}$ is a singular subspace of M. For $L \in \mathcal{L}$, we set $L^{\mathrm{t}} := \{A^{\mathrm{t}} \mid A \in L\}$, the image of L under transposition. The set $\mathcal{L}^{\mathrm{t}} = \{L^{\mathrm{t}} \mid L \in \mathcal{L}\}$ is the set of all maximal right ideals of M, see [Rei16, §2.3] for the complete classification. Every left ideal is the intersection of maximal left ideals, and analogously for right ideals.

By [Rei16, Thm. 2.38], the set $\mathcal{I} := \mathcal{L} \cup \mathcal{L}^{\mathrm{t}}$ is the set of all singular subspaces of M of maximal dimension. We will write $\mathcal{L}_W := \{L \cap W \mid L \in \mathcal{L}\}$ and $\mathcal{I}_W := \mathcal{L}_W \cup \mathcal{L}_W^{\mathrm{t}}$.

An important preliminary observation is the following proposition, which allows us to argue essentially as in [Rei16]:

8.3.5 Proposition. The map $\mathcal{I} \to \mathcal{I}_W$, $L \mapsto L \cap W$ is a bijection. Furthermore, $\mathcal{I}_W$ is the set of all singular subspaces of W of maximal dimension.

We require a sequence of lemmata which will be useful throughout the proof.

8.3.6 Lemma. Let $L \subseteq M$ be a linear space and assume that $S \in M$ satisfies $S \cdot L \subseteq W$. If $E_{ij} \in L$, then $S_{ji} = 0$.

Proof. Let $I := \{(i,j) \mid E_{ij} \in L\}$ and $\mathbb{C}[M] = \mathbb{C}[x_{ij} \mid 1 \leq i,j \leq d]$. We consider the matrix $A := \sum_{(i,j)\in I} x_{ij}E_{ij} \in \mathbb{C}[M]^{d\times d}$. Then, $0 = \operatorname{tr}(SA) = \sum_{i,j=1}^{d} S_{ji}A_{ij} = \sum_{(i,j)\in I} S_{ji}x_{ij}$ as a polynomial in $\mathbb{C}[M]$. Hence, we have $S_{ji} = 0$ for all $(i,j) \in I$ by comparing coefficients. ∎

8.3.7 Lemma. Let $L_1, L_2 \in \mathcal{L}$. Then, W does not contain $L_1 \cap L_2^{\mathrm{t}}$.

Proof. Assume for contradiction that $I := L_1 \cap L_2^{\mathrm{t}} \subseteq W$. By [Rei16, Korollar 2.28], there are $S, T \in \mathrm{GL}_d$ such that $L_1 \cdot T = L_2 \cdot S^{\mathrm{t}}$ are both equal to the space of all matrices with vanishing first column. Since L_1 is a left ideal and L_2^{t} a right ideal,

$$\begin{aligned} L &:= S \cdot I \cdot T = S \cdot (L_1 \cap L_2^{\mathrm{t}}) \cdot T = (S \cdot L_1 \cdot T) \cap (S \cdot L_2^{\mathrm{t}} \cdot T) \\ &= (L_1 \cdot T) \cap (S \cdot L_2^{\mathrm{t}}) = (L_1 \cdot T) \cap (L_2 \cdot S^{\mathrm{t}})^{\mathrm{t}} \end{aligned}$$

is equal to the space of matrices where the first row and the first column vanish. In particular, $E_{ij} \in L$ for all $i > 1$ and $j > 1$. Since conjugation leaves the trace invariant, $(ST)^{-1} \cdot L = T^{-1}IT \subseteq W$. By Lemma 8.3.6, this implies that $(ST)^{-1}$ has zero entries everywhere except for the first row and column. Hence, $\mathrm{rk}((ST)^{-1}) \leq 2 < 3 \leq d$, a contradiction because $(ST)^{-1} \in \mathrm{GL}_d$. ∎

8.3.8 Lemma. *W does not contain any nonzero left or right ideal of M.*

Proof. Since W is invariant under transposition, it suffices to show that W does not contain any left ideal of M. For contradiction, assume that $L \in \mathcal{L}$ satisfies $L \subseteq W$ and let r be the maximum rank among elements of L, then $r > 0$ because L is nonzero. By [Rei16, Satz 2.24], we have $\dim(L) = dr$ and by [FLR85, Theorem 2], a singular space with this property admits matrices $S, T \in \mathrm{GL}_d$ such that $L' := S \cdot L \cdot T$ is the space of all matrices whose first $d - r$ columns vanish. Because $r > 0$, we have $E_{id} \in L$ for all i. Since $(ST)^{-1} \cdot L' = T^{-1}LT \subseteq W$, Lemma 8.3.6 implies that the last row of $(ST)^{-1}$ vanishes, a contradiction. ∎

Proof of Proposition 8.3.5. By definition, the map $\mathcal{I} \to \mathcal{I}_W$, $L \mapsto L \cap W$ is surjective. To see that it is injective, assume for contradiction that there are $I_1, I_2 \in \mathcal{I}$ with $I_1 \neq I_2$ and

$$J := I_1 \cap W = I_2 \cap W.$$

By Lemma 8.3.8, we have $\dim(J) = d^2 - d - 1$. Furthermore, [Rei16, Lemma 2.29] implies $\dim(I_1 \cap I_2) \leq d^2 - 2d + 1$. Lemmata 8.3.7 and 8.3.8 yield that $I_1 \cap I_2$ is not contained in W, providing the last inequality in

$$d^2 - d - 1 = \dim(J) = \dim(I_1 \cap I_2 \cap W) \leq d^2 - 2d.$$

This means $d \leq 1$, which is the contradiction we sought.

Let $K \subseteq W$ be a singular subspace of maximal dimension. To finish the proof, we have to show that $K \in \mathcal{I}_W$.

Claim. There is some $L \in \mathcal{L}$ with $K \subseteq L$.

Proof of Claim. Note that $\dim(K) \geq d^2 - d - 1$ because K is of maximal dimension and for example, the singular spaces in $\mathcal{I}_W$ have dimension $d^2 - d - 1$. There are two cases to consider:

Case 1 ($d > 3$). In this case, $\dim(K) > d^2 - 2d + 2$. By [FLR85, Theorem 3], any singular space $K \subseteq M$ with this property is contained in a left or right ideal of M.

Case 2 ($d = 3$). In this case, we have $\dim(K) \geq 9 - 3 - 1 = 5$. Assume for contradiction that K is not contained in any maximal left or right ideal of M. As K is singular, it is contained in some maximal singular subspace of M, which we recall from Section 8.2. Since $\dim(K) > 3$ and K is not contained in any left or right ideal, K is contained in a space that is equivalent to

$$L := \left\{ \left(\begin{smallmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{smallmatrix} \right) \right\}.$$

Since $\dim(L) = 5 \leq \dim(K)$, in fact K must be equivalent to L. By definition, there are matrices $S, T \in \mathrm{GL}_d$ with $L = S \cdot K \cdot T$. It follows that $(ST)^{-1} \cdot L = T^{-1}KT \subseteq W$ and Lemma 8.3.6 implies that the last row and column of the invertible matrix $(ST)^{-1}$ vanish – a contradiction. □

It follows from the claim that $K \subseteq L \cap W$, the latter is a singular subspace of W. Therefore, $K = L \cap W$ because K is maximal, hence $K \in \mathcal{I}_W$. ■

With this at hand, we can mimic the six steps of the proof [Rei16, Thm. 4.2]. Recall that we denote by $t \in \mathrm{End}(W)$ the transposition operator $t(A) := A^{\mathrm{t}}$. The first step is the following:

8.3.9 Proposition. Let $g \in G_P$. After possibly replacing g by $g \circ t$, the following holds: For any $K \in \mathcal{L}_W$, we have $g(K) \in \mathcal{L}_W$.

Proof. We first note that for any $K \in \mathcal{I}_W$, we have $g(K) \in \mathcal{I}_W$ because g maps the singular subspaces of W to singular subspaces, and preserves the dimension.

Let $K_1, K_2 \in \mathcal{L}_W$ and $L_1, L_2 \in \mathcal{L}$ with $L_i \cap W = K_i$. After possibly composing g with t, we may assume that $g(K_1) \in \mathcal{L}_W$. Furthermore,

$$\begin{aligned}
\dim(g(K_1) \cap g(K_2)) &= \dim(g(K_1 \cap K_2)) && (g \text{ is injective}) \\
&= \dim(K_1 \cap K_2) && (g \text{ is injective}) \\
&= \dim(L_1 \cap L_2 \cap W) && (\text{by definition}) \\
&= \dim(L_1 \cap L_2) - 1 && (\text{Lemma 8.3.8}) \\
&= d^2 - 2d - 1. && [\text{Rei16, Lemma 2.29}]
\end{aligned}$$

If we had $g(K_2) \in \mathcal{L}_W^{\mathrm{t}}$, then Lemma 8.3.7 and [Rei16, Lemma 2.29] would imply that $\dim(g(K_1) \cap g(K_2)) = d^2 - 2d$. Hence, we have $g(K_2) \in \mathcal{L}_W$ as well. Since K_2 was arbitrary, the statement follows. ■

For the rest of the proof, fix some $g \in G_P$. We now construct two bijective maps $\phi, \psi \colon \mathbb{P}^{d-1} \to \mathbb{P}^{d-1}$ as follows: Note that g maps any $L(v) \cap W$ to some $L(w) \cap W$ by Proposition 8.3.9. Since $\mathcal{L}_W \cong \mathcal{L} \cong \mathbb{P}^{d-1}$, we can define ϕ and ψ via

$$g(L(v) \cap W) = L(\phi(v)) \cap W, \qquad g(L(v)^{\mathrm{t}} \cap W) = L(\psi(v))^{\mathrm{t}} \cap W.$$

As g is a bijection, so are ϕ and ψ. Moreover, ϕ and ψ are induced by semilinear maps $T\colon \mathbb{C}^d \to \mathbb{C}^d$ and $S\colon \mathbb{C}^d \to \mathbb{C}^d$, respectively. This follows from the upcoming Lemma 8.3.10 and the fundamental theorem of projective geometry [Rei16, Theorem 3.13].

8.3.10 Lemma. Both ϕ and ψ map projective lines to projective lines.

Proof. Assume that $u, v, w \in \mathbb{P}^{d-1}$ are three pairwise different lines contained in one plane, i.e., these projective points lie on the same projective line. Let $L := L(u) \cap L(v) \cap L(w)$. Then, $\dim(L) = d^2 - 2d$ by [Rei16, Satz 2.24, Korollar 2.27]. By Lemma 8.3.8, we have $\dim(L \cap W) = d^2 - d - 1$. Thus,

$$\begin{aligned} d^2 - 2d - 1 = \dim(g(L \cap W)) &= \dim\Big(a(L(v) \cap W) \cap g(L(u) \cap W) \cap g(L(w) \cap W)\Big) \\ &= \dim(L(\phi(u)) \cap L(\phi(v)) \cap L(\phi(w)) \cap W). \end{aligned}$$

Again by Lemma 8.3.8, we get that

$$d^2 - 2d = \dim(L(\phi(u)) \cap L(\phi(v)) \cap L(\phi(w)))$$

and [Rei16, Satz 2.24] implies that the lines $\phi(u)$, $\phi(v)$ and $\phi(w)$ span a plane. The argument for ψ is completely analogous. ■

We have shown that $g(L(v) \cap W) = L(Tv) \cap W$ and $g(L(v)^{\mathrm{t}} \cap W) = L(Sv)^{\mathrm{t}} \cap W$ for all v. The third step is to show that S and T are linear maps, so we have $S, T \in \mathrm{GL}_d$. The proof is word for word the same as in [Rei16, Thm 4.2,Proof Step 3], except that the matrix E_{11} has to be replaced with E_{13}, which is traceless. This is possible because we assumed $d \geq 3$.

Let $E := E_{11} + \cdots + E_{dd}$ be the identity matrix, so $M = W \oplus (\mathbb{C} \cdot E)$. We extend g to a map $\bar{g} \in \mathrm{GL}(M)$ by declaring $\bar{g}(E) := E$ and $\bar{g}|_W := g$. We then define the map

$$\begin{aligned} u\colon M &\longrightarrow M \\ A &\longmapsto S^{\mathrm{t}} \cdot \bar{g}(A) \cdot T \end{aligned}$$

Note that $u \in \mathrm{GL}(M)$ as well. For $A \in L(v) \cap W$, we know $g(A) \cdot Tv = 0$, so in particular we have $u(A)v = S^{\mathrm{t}} \cdot g(A) \cdot Tv = 0$, so $u(L(v) \cap W) \subseteq L(v)$. Equivalently, one has $u(L(v)^{\mathrm{t}} \cap W) \subseteq L(v)^{\mathrm{t}}$. We define

$$D_k := (E_{11} + E_{k1}) - (E_{1k} + E_{kk})$$

for $k > 1$, then $\{D_2, \ldots, D_d\} \cup \{E_{ij} \mid i \neq j\}$ is a basis of W. Certainly, $\mathbb{C}E_{ij}$ for $i \neq j$ can be expressed as an intersection of maximal left and right ideals of M. We observe that for $k > 2$, we can also write

$$\mathbb{C}D_k = \left(L(e_1 + e_k) \cap \bigcap_{i \notin \{1,k\}} L(e_i) \right) \cap \left(L(e_1 - e_k)^{\mathrm{t}} \cap \bigcap_{i \notin \{1,k\}} L(e_i)^{\mathrm{t}} \right).$$

It follows that $u(\mathbb{C}D_k) = \mathbb{C}D_k$ and $u(\mathbb{C}E_{ij}) = \mathbb{C}E_{ij}$ for $i \neq j$ and $k > 2$. Hence, there are certain $\mu_k, \mu_{ij} \in \mathbb{C}^\times$ with $u(D_k) = \mu_k D_k$ and $u(E_{ij}) = \mu_{ij} E_{ij}$. We will show that all these coefficients are identical:

8.3.11 Lemma. There is a $\mu \in \mathbb{C}^\times$ such that $\mu = \mu_k$ and $\mu = \mu_{ij}$ for all $i \neq j$ and $k > 2$.

Proof. For three distinct indices i, j, k (note that we require $d \geq 3$ here), the line spanned by $E_{ij} + E_{kj}$ is contained in W and can be expressed as the intersection of maximal left and right ideals. Hence, there is some $\mu_{ijk} \in \mathbb{C}^\times$ with

$$\mu_{ij}E_{ij} + \mu_{kj}E_{kj} = u(E_{ij} + E_{kj}) = \mu_{ijk} \cdot (E_{ij} + E_{kj}) = \mu_{ijk}E_{ij} + \mu_{ijk}E_{jk},$$

showing that $\mu_{ij} = \mu_{ijk} = \mu_{kj}$. Similarly, one can show that $\mu_{ij} = \mu_{ik}$. It follows that there is a $\mu \in \mathbb{C}^\times$ with $\mu = \mu_{ij}$ for all $i \neq j$. We are left to show that $\mu_k = \mu$ holds for all $k > 2$. For this purpose, let $j \notin \{1, k\}$ be a third index and observe that the line spanned by $D_k + E_{j1} - E_{jk}$ is contained in W and can be expressed as the intersection of maximal left and right ideals. Indeed, it is equal to

$$\left(L(e_1 + e_k) \cap \bigcap_{i \notin \{1,k\}} L(e_i) \right) \cap \left(L(e_1 - e_k)^{\mathrm{t}} \cap L(e_1 - e_j) \cap \bigcap_{i \notin \{1,j,k\}} L(e_i)^{\mathrm{t}} \right).$$

From this, we get certain $\nu_{jk} \in \mathbb{C}^\times$ with

$$\begin{aligned} \mu_k D_k + \mu E_{j1} - \mu E_{jk} &= u(D_k + E_{j1} - E_{jk}) = \nu_{jk} \cdot (D_k + E_{j1} - E_{jk}) \\ &= \nu_{jk} D_k + \nu_{jk} E_{j1} - \nu_{jk} E_{jk} \end{aligned}$$

and therefore, $\mu_k = \nu_{jk} = \mu$. ∎

We conclude that $u = \mu \cdot \mathrm{id}_M$. We can replace S by $\mu^{-1}S$ and achieve that u is the identity. Consequently, we are done by Lemma 8.3.4.

8.4 The Boundary of the 4×4 Determinant

The next logical step would be to study $\det_4$ and $\det_5$ to accumulate a more solid foundation of examples. The maximal linear subspaces of $Z(\det_4)$ up to action of the stabilizer are known [EH88, Cor. 1.3], [FLR85], they are as follows:

$$C_0 = \left\{ \begin{pmatrix} 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \right\} \text{ and}$$
$$C_1 = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\}$$

are compression spaces. The semistable spaces are the following:

$$L_1 = \left\{ \begin{pmatrix} 0 & \alpha & -\beta & * \\ -\alpha & 0 & \gamma & * \\ \beta & -\gamma & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix} \middle| \alpha, \beta, \gamma \in \mathbb{C} \right\},$$

$$L_2 = \left\{ \begin{pmatrix} \gamma & \delta & 0 & 0 \\ 0 & 0 & \gamma & \delta \\ -\alpha & 0 & -\beta & 0 \\ 0 & -\alpha & 0 & -\beta \end{pmatrix} \middle| \alpha, \beta, \gamma, \delta \in \mathbb{C} \right\},$$

$$L_3 = \left\{ \begin{pmatrix} -\beta & -\delta & 0 & 0 \\ \alpha & 0 & -\gamma & -\delta \\ -\delta & 0 & \beta & 0 \\ \gamma & \alpha & 0 & \beta \end{pmatrix} \middle| \alpha, \beta, \gamma, \delta \in \mathbb{C} \right\}.$$

By the computation given in Program 8.3, one can obtain the following result:

8.4.1 Proposition. For $A, B \in W$ we let $\langle A, B \rangle := \operatorname{tr}(A^t B)$. With respect to this bilinear form, we choose orthogonal complements $W = L_i \oplus K_i$ for $i \in \{1, 2\}$.

There are surjective linear maps $a_i \in \operatorname{End}(W, L_i)$ and $b_i \in \operatorname{End}(W, K_i)$ such that $a_i + b_i \cdot t$ is an approximation path from $\det_4$ to a polynomial $Q_i \in \partial\Omega(\det_4)$ and $\overline{\Omega}(Q_i)$ is a component of $\partial\Omega(\det_4)$ for $i \in \{1, 2\}$. ■

Remark. The polynomial Q_1 is equal to

$$\begin{aligned} Q_1 &= \operatorname{tr}\left(\begin{pmatrix} 0 & x_1 & -x_2 & z_3 \\ -x_1 & 0 & x_3 & z_2 \\ x_2 & -x_3 & 0 & z_1 \\ 0 & 0 & 0 & t \end{pmatrix}^{\sharp} \cdot \begin{pmatrix} c & e & f & 0 \\ e & b & d & 0 \\ f & d & a & 0 \\ y_3 & y_2 & y_1 & 0 \end{pmatrix} \right) \\ &= t \cdot (a \cdot x_1^2 + b \cdot x_2^2 + c \cdot x_3^2 + d \cdot x_1 x_2 + e \cdot x_2 x_3 + f \cdot x_1 x_3) \\ &\quad - (y_1 x_1 + y_2 x_2 + y_3 x_3)(z_1 x_1 + z_2 x_2 + z_3 x_3), \end{aligned}$$

as a polynomial in $\mathbb{C}[t, a, b, c, d, e, f, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3]$. Note that the first polynomial from Theorem 8.0.1 appears with the factor t as the first summand here. The polynomial Q_2 is equal to

$$\begin{aligned} Q_2 &= \operatorname{tr}\left(\begin{pmatrix} x_1 & x_2 & 0 & 0 \\ 0 & 0 & x_1 & x_2 \\ x_3 & 0 & x_4 & 0 \\ 0 & x_3 & 0 & x_4 \end{pmatrix}^{\sharp} \cdot \begin{pmatrix} y_1 & y_2 & b_1 & b_2 \\ b_3 & b_4 & -y_1 & -y_2 \\ y_3 & a_1 & y_4 & a_2 \\ a_3 & -y_3 & a_4 & -y_4 \end{pmatrix} \right) \\ &= x_1^2(x_3 a_2 - x_4 a_1) + x_2^2(x_3 a_4 - x_4 a_3) + x_3^2(b_1 x_2 - b_2 x_1) + x_4^2(b_4 x_1 - b_3 x_2) \\ &\quad - (y_1 x_2 x_3 x_4 - x_1 y_2 x_3 x_4 + x_1 x_2 y_3 x_4 - x_1 x_2 x_3 y_4), \end{aligned}$$

as a polynomial in $\mathbb{C}[x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4]$. ○

Remark. As mentioned in [Lan15], these components were first observed by J. Brown, N. Bushek, L. Oeding, D. Torrance and Y. Qi. ○

Remark. We used several approximation paths $a + bt$ for $\det_4$ with $a \in \operatorname{End}(W, L_3)$. Every such path gave a degeneration Q of $\det_4$ whose orbit closure $\overline{\Omega}_Q$ had codimension one in $\partial\Omega(\det_4)$, i.e., $\overline{\Omega}_Q$ was not a component of the boundary.

```python

from sympy import *
a,b,c,d,e,f,p,q,r,s,u,v,w,x,y,z = X = symbols(
 ' '.join(c for c in 'abcdefpqrsuvwxyz'))

def OrbitDimension(P,X):
    P = Poly(P,X)
    Df = [ Poly(x)*diff(P,y) for x in X for y in X ]
    Mn = list(set(sum((Poly(Q).monoms() for Q in Df),[])))
    Mx = Matrix([[ Q.coeff_monomial(m) for Q in Df ]
       for m in Mn ])
    return Mx.rank()

L1 = Matrix([
 [ 0, x,-y, p ],
 [-x, 0, z, q ],
 [ y,-z, 0, r ],
 [ 0, 0, 0, s ] ])
K1 = Matrix([
 [ c, e, f, 0 ],
 [ e, b, d, 0 ],
 [ f, d, a, 0 ],
 [ u, v, w, 0 ] ])
Q1 = (L1.adjugate()*K1).trace()

L2 = Matrix([
 [ p, q, 0, 0 ],
 [ 0, 0, p, q ],
 [ r, 0, s, 0 ],
 [ 0, r, 0, s ] ])
K2 = Matrix([
 [ a, b, c, d ],
 [ e, f,-a,-b ],
 [ u, v, w, x ],
 [ y,-u, z,-w ] ])
Q2 = (L2.adjugate()*K2).trace()

for Q in (Q1, Q2):
 if OrbitDimension(Q,X)==(16*16)-(2*4*4-2)-1:
  print("Found Component:\n%s" % str(Q.expand()))
```

Program 8.3: Computing degenerations of $\det_4$ with *Python* [Pyt; Sym].

We note an interesting fact about the geometry of the spaces $E_i := G \circ \mathrm{End}(W, L_i)$ where $G := G_{\det_4}$, which might be related. From the proof of [EH88, Thm. 1.2], one can deduce that $(\overline{E_2})^{\mathrm{ss}}$ is the disjoint union of E_2^{ss} and E_3^{ss}. In other words, E_3^{ss} is the complement of E_2^{ss} in its closure, its "boundary" so to speak. ○

This leaves the following question:

8.4.2 Question. Does $\partial\Omega(\det_4)$ contain any irreducible component other than the orbit closures of Q_1, Q_2 and the traceless determinant?

It is also noteworthy that L_1 has the space of skew-symmetric 3×3 matrices as its "primitive part", see [EH88] for a definition. We also ask if the construction of this component generalizes:

8.4.3 Question. Let $d = 2k \in \mathbb{N}$. Let $W := \mathbb{C}^{d \times d}$ and consider the linear subspace

$$L := \left\{ \left(\begin{array}{ccc|c} & & & * \\ & A & & \vdots \\ & & & * \\ \hline 0 & \cdots & 0 & * \end{array} \right) \;\middle|\; \begin{array}{l} A \in \mathbb{C}^{(d-1)\times(d-1)}, \\ A = -A^{\mathrm{t}} \end{array} \right\}$$

Clearly, L is a singular subspace of $\det_d$.

We ask whether there exists a vector space complement $W = L \oplus U$ and linear maps $a\colon W \twoheadrightarrow L$ and $b\colon W \twoheadrightarrow U$ such that $a + bt$ is an approximation path from $\det_d$ to a polynomial Q whose orbit closure is an irreducible component of $\partial\Omega(\det_d)$.

We end with the sobering remark that the 5×5 case is already quite substantially more difficult. The annihilator $\mathcal{A}_{\det_5}$ contains infinitely many orbits under the action of $G_{\det_d}$, as shown in [Bor+16, Theorem 5]. This means that for general d, the $G_{\det_d}$-orbit structure of $\mathcal{A}_{\det_d}$ is presumably quite involved. Understanding this structure is more or less equivalent to the classification of all maximal linear subspaces of $\det_d$.

Chapter 9

The Binomial

It would be desirable to have a description of $\partial\Omega(P_d)$ for a complete family of polynomials $P = (P_d)_{d\in\mathbb{N}}$. A trivial example is $(\mathrm{mn}_d)_{d\in\mathbb{N}}$ because it is closed, as noted in Lemma 4.3.1. However, this is of little educational value. Our suggestion is to study the polynomial family

$$\mathrm{bn}_d := x_1 \cdots x_d + y_1 \cdots y_d, \tag{1}$$

the generic **binomial**. We assume $d \geq 3$ throughout.

9.0.1 Remark. Note that $\mathrm{bn}_d \in \overline{\Omega}(\det_d)$. For example,

$$\mathrm{bn}_5 = \det \begin{pmatrix} x_1 & y_1 & 0 & 0 & 0 \\ 0 & x_2 & y_2 & 0 & 0 \\ 0 & 0 & x_3 & y_3 & 0 \\ 0 & 0 & 0 & x_4 & y_4 \\ y_5 & 0 & 0 & 0 & x_5 \end{pmatrix}$$

and this construction generalizes easily.

Furthermore, bn_d is also a special case of another important polynomial family, namely the sums of products polynomial $\sum_{i=1}^{k}\prod_{j=1}^{d} x_{kj}$, see [CKW10, Chapter 11] and [Kay12]. ●

9.0.2. Here is a brief summary of our analysis of the binomial:

(1) Let $\mathcal{S} \subseteq \mathrm{End}(W)$ be the set of noninvertible endomorphisms. The set $B := \overline{\mathrm{bn}_d \circ \mathcal{S}}$ is an irreducible component of $\partial\Omega(\mathrm{bn}_d)$. It is not an orbit closure.

(2) There is only one semistable linear subspace of $Z(\mathrm{bn}_d)$. Using it, we construct a linear degeneration Q of bn_d such that $\overline{\Omega}_Q$ is an irreducible component of the boundary.

(3) We prove that the scheme $\hat{\mathcal{A}}_{\mathrm{bn}_d}$ is generically smooth, in particular generically reduced. If it is reduced and its components intersect transversally, then $\partial\Omega(\mathrm{bn}_d)$ has exactly these two irreducible components. ●

We let $W := \mathbb{C}^d \times \mathbb{C}^d$ with coordinates x_i on the left factor and y_j on the right one, so we have $\mathrm{bn}_d \in V := \mathbb{C}[W]_d$. It is easy to see that bn_d is concise.

9.1 Stabilizer and Maximal Linear Subspaces

We will denote by $H_d := G_{\mathrm{bn}_d}$ the stabilizer group of the binomial. The main goal of this section is to prove the following result:

9.1.1 Theorem. Let $W := \mathbb{C}^d \times \mathbb{C}^d$ with $d \geq 3$. The stabilizer group $H_d \subseteq \mathrm{GL}(W)$ is reductive of dimension $2d-2$. Moreover,

(1) The identity component of the stabilizer of P is the group

$$H_d^\circ = \left\{ \mathrm{diag}(s_1, \dots, s_d, t_1, \dots, t_d) \;\middle|\; \prod_{i=1}^d s_i = \prod_{i=1}^d t_i = 1 \right\}.$$

(2) Let $K \subseteq \mathrm{GL}(W)$ be the group generated by

- the map $t\colon W \to W$, $(v, w) \mapsto (w, v)$, and
- the group $\mathfrak{S}_d \times \mathfrak{S}_d \subseteq \mathrm{GL}_d \times \mathrm{GL}_d \subseteq \mathrm{GL}(W)$ of permutation matrices that permute the first and last d coordinates among themselves, respectively.

Then, $K \cong \mathfrak{S}_d \wr \mathbb{Z}_2$ and $H_d = K \cdot H_d^\circ$ has $|K| = 2 \cdot d! \cdot d!$ connected components.

9.1.2 Corollary. By Theorem A.1.9, $\dim(\overline{\Omega}(\mathrm{bn}_d)) = 4d^2 - 2d + 2$. ■

One can prove Theorem 9.1.1 by means of the explicit description of $\mathfrak{Lie}(H_d)$ from Corollary 7.2.3, but another method is to study the action of H_d on linear subspaces of the vanishing locus $\mathrm{Z}(\mathrm{bn}_d) \subseteq W = \mathbb{C}^d \times \mathbb{C}^d$. It is straightforward to provide linear subspaces of $\mathrm{Z}(\mathrm{bn}_d)$: The most canonical choice are the spaces $L_{ij} := \mathrm{Z}(x_i, y_j)$ for $1 \leq i, j \leq d$. Furthermore, there is the space

$$L_0 := \{ (w_1, \dots, w_d, \zeta w_1, \dots, \zeta w_d) \mid w_1, \dots, w_d \in \mathbb{C} \} \subseteq W,$$

where $\zeta \in \mathbb{C}$ will always denote a fixed d-th root of -1, i.e., we have $\zeta^d = -1$. For the proof of Theorem 9.1.1, we will prove the following proposition as an auxiliary result.

9.1.3 Proposition. The maximal linear subspaces of $\mathrm{Z}(\mathrm{bn}_d)$ consist of the L_{ij} and the spaces $h(L_0)$ for $h \in H_d$. Furthermore, the L_{ij} are unstable.

Proof of Theorem 9.1.1. It is easy to see that the elements of H_d° and K stabilize bn_d. We are left to show that $H_d \subseteq K \cdot H_d^\circ$. The elements of H_d act on the set of linear subspaces of $\mathrm{Z}(\mathrm{bn}_d)$ of any fixed dimension, so by Proposition 9.1.3 they permute the L_{ij}. Dually, the action of H_d on W^* permutes the spaces $L_{ij}^* := \langle x_i, y_j \rangle \subseteq W^*$ for all $1 \leq i, j \leq d$ in the sense that for all $h \in H_d$, there are $1 \leq r, s \leq d$ such that

$$L_{ij}^* \circ h := \langle x_i \circ h, y_j \circ h \rangle = \langle x_r, y_s \rangle = L_{rs}^*.$$

Let $h \in H_d$ and $1 \le k \le d$. We claim that there is an index $1 \le i \le d$ such that the linear form $x_k \circ h$ is either a scalar multiple of x_i or y_i. To this end, let $1 \le i, j, r, s \le d$ be such that $L^*_{k1} \circ h = L^*_{ij}$ and $L^*_{k2} \circ h = L^*_{rs}$, then

$$\langle x_k \circ h \rangle = \langle x_k \rangle \circ h = (L^*_{k1} \cap L^*_{k2}) \circ h = L^*_{ij} \cap L^*_{rs}.$$

Hence, the two planes L^*_{ij} and L^*_{rs} intersect in the line spanned by $(x_k \circ h)$, in particular this intersection is nontrivial. It follows that $i = r$ or $j = s$ because for $i \ne r$ and $j \ne s$, we have $L^*_{ij} \cap L^*_{rs} = \{0\}$. Therefore, either $\langle x_k \circ h \rangle = \langle x_i \rangle$ or $\langle x_k \circ h \rangle = \langle y_j \rangle$ as claimed.

In the same manner, we can show that $y_k \circ h$ is some scalar multiple of a coordinate function. This means that h is the product of a permutation and a diagonal matrix. The result is a straightforward corollary. ■

We are left to show Proposition 9.1.3. The proof has been split up into the following three lemmata:

9.1.4 Lemma. A maximal linear subspace $L \subseteq \mathrm{Z}(\mathrm{bn}_d)$ which is contained in a coordinate hyperplane is equal to L_{ij} for some choice of i and j.

Proof. Assume $L \subseteq \mathrm{Z}(x_i)$. It follows that any $(v_1, \ldots, v_d, w_1, \ldots, w_d) \in L$ must satisfy $w_1 \cdots w_d = 0$. Consequently, $L \subseteq L_{i1} \cup \cdots \cup L_{id}$. It follows that $L = L_{ij}$ for some j. ■

9.1.5 Lemma. The spaces L_{ij} are unstable.

Proof. It is sufficient to show that L_{1d} is unstable since $L_{ij} = h(L_{1d})$ for an appropriate permutation $h \in H_d$. To see that L_{1d} is unstable, let $a \colon W \to L_{1d}$ be any linear map and consider

$$h_\varepsilon := \operatorname{diag}(\varepsilon^{1-d}, \varepsilon, \ldots, \varepsilon, \varepsilon^{1-d}) \in H_d,$$

where ε^{1-d} is in the first and last position. Note that the representing matrix of a has vanishing first and last row. Therefore, $h_\varepsilon \circ a \to 0$ as $\varepsilon \to 0$. ■

9.1.6 Lemma. Let $L \subseteq \mathrm{Z}(\mathrm{bn}_d)$ be a maximal linear subspace which is not contained in any coordinate hyperplane. Then, there is an $h \in H_d$ such that $h(L) = L_0$.

Proof. Since L is not contained in any of the coordinate hyperplanes, there is a point $(u_1, \ldots, u_d, v_1, \ldots, v_d) \in L$ such that $u_i \ne 0$ and $v_i \ne 0$ for all $1 \le i \le d$. Let $u \in \mathbb{C}$ be such that $u^d = u_1 \cdots u_d$. We apply the scaling matrix

$$\operatorname{diag}\left(\frac{u}{u_1}, \ldots, \frac{u}{u_d}, \frac{v_2 \cdots v_d}{\zeta^{d-1} u^{d-1}}, \frac{\zeta u}{v_2}, \ldots, \frac{\zeta u}{v_d}\right) \in H_d^\circ$$

to the space L and thereby achieve that $(u, \ldots, u, \zeta u, \ldots, \zeta u) \in L$. By scaling, we get that $p := (1, \ldots, 1, \zeta, \ldots, \zeta) \in L$. We denote by K the discrete part of H_d, i.e., the finite group of permutations that stabilize bn_d. We now claim that $L \subseteq K.L_0$, so L is contained in a finite union of translates of L_0. This will imply the statement because L has to be equal to one of the translates of L_0. For the proof, let $q = (u_1, \ldots, u_d, v_1, \ldots, v_d) \in L$ be any point different from p. Since the entire line containing p and q must be a subset of L and subsequently a subset of $\mathrm{Z}(\mathrm{bn}_d)$, this means that bn_d vanishes at $Tp - q$ for any $T \in \mathbb{C}$. In other words,

$$0 = \prod_{i=1}^{d}(T - u_i) + \prod_{i=1}^{d}(\zeta T - v_i) = \prod_{i=1}^{d}(T - u_i) - \prod_{i=1}^{d}(T - \zeta^{-1} v_i)$$

as an identity of polynomials in $\mathbb{C}[T]$. Hence, $u_i = \zeta^{-1} v_i$ up to a permutation which stabilizes bn_d. ■

For future reference we consider the matrix

$$b := \begin{pmatrix} \mathbb{I} & 0 \\ \zeta\mathbb{I} & 0 \end{pmatrix}, \tag{2}$$

then $\mathrm{End}(W, L_0) = \overline{b \circ \mathrm{GL}(W)}$ is the set of all endomorphisms that map into L_0. From Proposition 9.1.3, we obtain:

$$\mathcal{A}^{\mathrm{ss}}_{\mathrm{bn}_d} = \{h \circ a \mid a \in \mathrm{End}(W, L_0)^{\mathrm{ss}}\} = H_d \circ \mathrm{End}(W, L_0)^{\mathrm{ss}} = \overline{H_d \circ b \circ \mathrm{GL}(W)}^{\mathrm{ss}}. \tag{3}$$

9.2 The First Boundary Component

There are certain natural degenerations of bn_d which we will study here. We introduce the notation $P \sim Q :\Leftrightarrow P \in \Omega_Q$ and say that P is **linearly equivalent** to Q in this case.

9.2.1 Remark. An important property of linear equivalence is the fact that $P \sim Q$ implies $G_P \cong G_Q$, indeed if $Q = P \circ g$ then it is easy to see that $G_Q = g^{-1} \circ G_P \circ g$. ●

For the rest of this section, we also set

$$\mathcal{S} := \{a \in \mathrm{End}(W) \mid \mathrm{rk}(a) < 2d\},$$

so $\mathcal{S}$ is the hypersurface of noninvertible endomorphisms on $W = \mathbb{C}^{2d}$. Our main result of this section is the following:

9.2.2 Proposition. Let $B \subseteq \overline{\Omega}(\mathrm{bn}_d)$ be the closure of $\mathrm{bn}_d \circ \mathcal{S}$. Then, B has codimension one in $\overline{\Omega}(\mathrm{bn}_d)$ and it is an irreducible component of $\partial\Omega(\mathrm{bn}_d)$.

For the proof, we define the polynomials

$$Q_{r,s} = x_1 \cdots x_d + y_1 \cdots y_{d-1} \cdot \left(\sum_{k=1}^{r} x_k + \sum_{k=1}^{s} y_k \right)$$

for $r \leq d$ and $s < d$. Note that $Q_{r,s} \in \mathrm{bn}_d \circ \mathcal{S}$. We defer the proof of the following auxiliary result:

9.2.3 Lemma. The polynomials $Q_{d-1,d-1}$ and $Q_{d,d-1}$ are elements of $\mathrm{bn}_d \circ \mathcal{S}$ whose orbits are disjoint. Furthermore, both orbits are of codimension 2 in $\overline{\Omega}(\mathrm{bn}_d)$.

Proof of Proposition 9.2.2. By definition, B is an irreducible closed subset of $\overline{\Omega}(\mathrm{bn}_d)$, because it is the closure of the image of the irreducible variety $\mathcal{S}$ under the orbit map. Let us check that it is contained in $\partial\Omega(\mathrm{bn}_d)$: The elements of $\mathrm{bn}_d \circ \mathcal{S}$ are not concise. Since that is a closed condition by Remark 5.1.3, no element of B is concise. As all elements of $\Omega(\mathrm{bn}_d)$ are concise, we know that B must lie in the complement.

We are left to determine the dimension of B. By Lemma 9.2.3, B has codimension at most 2. Assume for contradiction that the codimension of B in $\partial\Omega(\mathrm{bn}_d)$ is equal to 2. The orbit closure $\overline{\Omega}(Q_{d,d-1})$ is an irreducible closed subset of B and it has the same dimension as B by Lemma 9.2.3 and our assumption. This implies that $\overline{\Omega}(Q_{d,d-1}) = B$. We have $Q_{d-1,d-1} \in B = \overline{\Omega}(Q_{d,d-1})$, but $Q_{d-1,d-1} \notin \Omega(Q_{d,d-1})$ by Lemma 9.2.3. This means $Q_{d-1,d-1} \in \partial\Omega(Q_{d,d-1})$. However, the orbit of $Q_{d-1,d-1}$ has the same dimension as the orbit of $Q_{d,d-1}$, which is a contradiction to Theorem A.1.9.(3).

It follows that the codimension of B is at most 1, and since it is completely contained in the boundary, this implies that it is a component. ■

Proof of Lemma 9.2.3. Let $r \in \{d-1, d\}$ and $Q := Q_{r,d-1}$. We define the linear form $\ell := x_1 + \cdots + x_r + y_1 + \cdots + y_{d-1}$, so $Q = x_1 \cdots x_d + y_1 \cdots y_{d-1} \cdot \ell$. Furthermore, we denote by $a \in \mathcal{S}$ the endomorphism that satisfies $y_d \circ a = \ell$ and a is the identity on all other coordinates, so that $Q = \mathrm{bn}_d \circ a$. We will compute the dimension of G_Q and the number of its connected components. We will see that the former does not depend on r, whereas the latter does – this will imply the lemma by Remark 9.2.1. To obtain a description of G_Q, we study its action on the set of linear subspaces of $\mathrm{Z}(Q)$.

Claim. If $L \subseteq \mathrm{Z}(Q)$ is a linear subspace with $\dim(L) = 2d-2$, then there are indices $1 \le i, j \le d$ such that $L = \mathrm{Z}(x_i, y_j \circ a)$.

For $j < d$, this means $L = L_{ij} = \mathrm{Z}(x_i, y_j)$ and otherwise $L = \mathrm{Z}(x_i, \ell)$.

Proof of Claim. Since $Q = \mathrm{bn}_d \circ a$, we know that $a(L)$ is a linear subspace of $\mathrm{Z}(\mathrm{bn}_d)$, so the space $a(L)$ is contained in one of the spaces from Proposition 9.1.3. We first show that there are $1 \le i, j \le d$ with $a(L) \subseteq L_{ij}$.

The form $y_d - \ell$ vanishes on $\mathrm{im}(a)$, but it is straightforward to verify that it does not vanish on $h(L_0)$ for any $h \in H_d$. Therefore, $a(L) \neq L_0$, but it might be properly contained in it. We show that this is not possible: Since $d \ge 3$, we have

$$\dim(a(L)) \ge \dim(L) - 1 = 2d - 3 \ge d = \dim(L_0).$$

Hence, we have $a(L) \subseteq L_{ij}$. In particular, the linear forms $x_i = x_i \circ a$ and $y_j \circ a$ both vanish on L. The latter is equal to y_j for $j < d$ and otherwise it is equal to ℓ. In either case however, the two forms are linearly independent and therefore cut out the space L. □

The action of G_Q permutes these linear spaces. Dually, its action on W^* permutes the spaces $\langle x_i, y_j \circ a\rangle \subseteq W^*$. Reminiscent of the proof of Theorem 9.1.1, we can conclude in the same fashion that the action of G_Q maps every element of the set $M := \{x_1, \ldots, x_d, y_1, \ldots, y_{d-1}, \ell\}$ to a scalar multiple of an element of M. Of course, it can map y_d to any linear form which is linearly independent from the other coordinates. Hence, there is a group $K \subseteq \mathrm{GL}_{2d-1}$ such that G_Q admits a matrix representation

$$G_Q = \left\{ \left(\begin{array}{ccc|c} & & & 0 \\ & h & & \vdots \\ & & & 0 \\ \hline * & \cdots & * & \alpha \end{array} \right) \;\middle|\; \begin{array}{l} h \in K, \\ \alpha \in \mathbb{C}^\times \end{array} \right\} \cong K \times \mathbb{C}^{2d-1} \times \mathbb{C}^\times. \tag{4}$$

Claim. K is a finite group of order $|K| = r! \cdot d^2 \cdot (d-1)!$.

This claim implies the lemma as follows: We have $\dim(G_Q) = 2d = \dim(H_d) + 2$, the latter by Theorem 9.1.1. From Theorem A.1.9, we conclude that Ω_Q has codimension two in $\overline{\Omega}(\mathrm{bn}_d)$. If $Q_{d-1,d-1}$ and $Q_{d,d-1}$ were in the same orbit, then their stabilizers would be conjugate and in particular, they would both have the same number of connected components. However, the number of connected components of G_Q is equal to $|K|$, which depends on r.

We are therefore left to verify our claim. We first study the subgroup

$$\Pi := \{\pi \in K \mid \ell \circ \pi \in \mathbb{C}^\times \ell\}$$

of K and observe that its order depends on r:

Claim. We have $\Pi \cong \mathbb{Z}_d \rtimes (\mathfrak{S}_r \times \mathfrak{S}_{d-1})$, where $\mathbb{Z}_d$ corresponds to scaling all variables with a d-th root of unity simultaneously and $\mathfrak{S}_r \times \mathfrak{S}_{d-1}$ permutes the x-variables and the y-variables that occur in ℓ among themselves.

Proof of Claim. We first check that every $\pi \in \Pi$ is a product of a diagonal and a permutation matrix. Let $\mathbb{M} := \{\mathbb{C}x_1, \ldots, \mathbb{C}x_d, \mathbb{C}y_1, \ldots, \mathbb{C}y_{d-1}, \mathbb{C}\ell\}$ be the set of lines spanned by elements of M. We know that π induces a permtutation of $\mathbb{M}$ which has $\mathbb{C}\ell$ as a fixpoint. In particular, π induces a permutation of the lines spanned by the first $2d-1$ coordinates. This is precisely what we claimed.

Next, we show that all diagonal matrices in Π are scalar and correspond to multiplication with a d-th root of unity. Let $\pi \in \Pi$ be a diagonal matrix and let $\zeta \in \mathbb{C}^\times$ be such that $\ell \circ \pi = \zeta\ell$. It is straightforward to see that this implies $x_i \circ \pi = \zeta x_i$ and $y_j \circ \pi = \zeta y_j$ for all $1 \leq i, j \leq d-1$. As π stabilizes

$$Q = x_1 \cdots x_d + y_1 \cdots y_{d-1} \cdot (x_1 + \cdots + x_r + y_1 + \cdots + y_{d-1}),$$

it must, for instance, stabilize the monomial $y_1^2 \cdot y_2 \cdots y_d$, whence $\xi^d = 1$. Let $\vartheta \in \mathbb{C}^\times$ be such that $x_d \circ \pi = \vartheta x_d$. As π also stabilizes the monomial $x_1 \cdots x_d$, we also get $\vartheta \xi^{d-1} = 1$, hence $\vartheta = \xi$.

We now assume that $\pi \in \Pi$ is a permutation and show that π does not map any x-variable to a y-variable. Observe that any monomial of Q that is divisible by a y-variable is divisible by $d-1$ of the y-variables. Assume for contradiction that some x-variable is mapped to a y-variable. Then, since $(x_1 \cdots x_d) \circ \pi = \prod_{i=1}^d (x_i \circ \pi)$ is a monomial of Q, we conclude that $d-1$ of the d x-variables must be mapped to y-variables. In turn, all y-variables must be mapped to x-variables. As Q contains the monomial $y_1^2 \cdot y_2 \cdots y_{d-1}$, this implies the contradiction that $Q = Q \circ \pi$ also contains the monomial $x_1^2 \cdot x_2 \cdots x_{d-1}$.

We are left to verify that for $r = d-1$, every permutation $\pi \in \Pi$ leaves x_d invariant. Assume for contradiction that there is some $i \le d-1$ such that $x_i \circ \pi = x_d$. Then, $Q = Q \circ \pi$ contains the monomial $(y_1 \cdots y_{d-1} \cdot x_i) \circ \pi = y_1 \cdots y_{d-1} \cdot x_d$, a contradiction. □

As $|\Pi| = r! \cdot d \cdot (d-1)!$, we are left to verify that K/Π contains d residue classes, by Lagrange's Theorem. For any $1 \le j < d$, denote by $g_j \in K$ the map that satisfies $y_j \circ g_j = -\ell$ and is the identity on all other variables. Hence, $\ell \circ g_j = -y_j$ and it is easy to check that g_j stabilizes Q. We will show that

$$K/\Pi = \{\Pi, g_1\Pi, \ldots, g_{d-1}\Pi\}\,. \tag{5}$$

To this end, let $g \in K$ be such that $\ell \circ g$ is not a scalar multiple of ℓ. Then, there is some $\beta \in \mathbb{C}^\times$ with $\ell \circ g = \beta \cdot z$, where z is a variable. Furthermore, there is an $\eta \in \mathbb{C}^\times$ such that g maps a variable $\tilde{z} \in M$ to $\eta \cdot \ell$. By applying an element of the subgroup Π, we may assume that $z = \tilde{z}$, i.e., we have $z \circ g = \eta \cdot \ell$. We may furthermore assume that g maps all variables other than z to scalar multiples of themselves. By studying the monomials of Q, it is straightforward to check that $z = y_j$ for some $1 \le j < d$. There are certain scalars α_i and β_k such that

$$\beta \cdot y_j = \ell \circ g = \left(\textstyle\sum_{i=1}^r x_i + \sum_{k=1}^{d-1} y_k\right) \circ g = \left(\textstyle\sum_{i=1}^r \alpha_i x_i\right) + \left(\textstyle\sum_{k \neq j} \beta_k y_k\right) + \eta \cdot \ell.$$

With $\beta_j := -\beta$, we obtain that $-\eta \cdot \ell = \sum_{i=1}^r \alpha_i x_i + \sum_{k=1}^{d-1} \beta_k y_k$, so $\alpha_i = -\eta$ and $\beta_k = -\eta$ for all i and k. In particular, $\eta = \beta$. It is now again easy to check that $-\eta$ is a d-th root of unity, so we have achieved $g \in g_j K$ and proved (5). ■

It is natural to ask what a generic elements of $B := \overline{\mathrm{bn}_d \circ \mathcal{S}}$ looks like. It turns out that B is not an orbit closure. Instead, for the one-parameter family of polynomials

$$\hat{Q}_t := t \cdot x_1 \cdots x_d + y_1 \cdots y_{d-1} \cdot \left(\sum_{k=1}^{d} x_k + \sum_{k=1}^{d-1} y_k \right), \qquad (t \in \mathbb{C}^\times)$$

we prove the following result.

9.2.4 Proposition. Let $B := \overline{\mathrm{bn}_d \circ \mathcal{S}}$. The union $U := \bigcup_{t\in\mathbb{C}^\times} \Omega(\hat{Q}_t)$ is dense in B and contains an open subset of B. In particular, a generic element of B is linearly equivalent to $\hat{Q}_t$ for some $t \in \mathbb{C}^\times$.

For the proof, we require the following technical lemma whose proof we postpone until after the proof of Proposition 9.2.4.

9.2.5 Lemma. Let $a \in \mathcal{S}$ with $\dim(\ker(a)) = 1$ and $P := \mathrm{bn}_d \circ a$. Then, one of the following is true:

(1) There is some $t \in \mathbb{C}^\times$ such that $P \sim \hat{Q}_t$.

(2) There are natural numbers $r \le d$ and $s \le d-1$ such that $P \sim Q_{r,s}$.

Furthermore, the codimension of $\overline{\Omega}_P$ in B is at least 1, so it is a proper closed subset of B.

Proof of Proposition 9.2.4. The map $\omega\colon \mathcal{S} \to B$, $a \mapsto \mathrm{bn}_d \circ a$ is a dominant morphism. The set $\mathcal{S}_1 := \{a \in \mathcal{S} \mid \dim(\ker(a)) = 1\}$ is open and dense in $\mathcal{S}$. Since ω is dominant, this means that $\omega(\mathcal{S}_1)$ is dense and contains a nonempty open subset, the latter by [TY05, 15.4.3]. By Lemma 9.2.5, we have $\omega(\mathcal{S}_1) \setminus Z = U$, where

$$Z := \bigcup\nolimits_{r=1}^{d} \bigcup\nolimits_{s=1}^{d-1} \overline{\Omega}(Q_{r,s}).$$

By Lemma 9.2.5, Z is a finite union of proper closed subsets of B, therefore Z itself is a proper closed subset of B. Consequently, U is dense and contains a nonempty open subset. ∎

9.2.6 Remark. Of course, it is equally true that the set

$$\bigcup_{t\in\mathbb{C}^\times} \Omega(\hat{Q}_t) \cup \bigcup_{\substack{1\le r\le d\\ 1\le s<d}} \Omega(Q_{r,s})$$

is dense in B and contains an open subset, the statement of Proposition 9.2.4 merely emphasizes the fact that the $Q_{r,s}$ are just a finite number of "special" cases. ●

Proof of Lemma 9.2.5. By Lemma 8.3.3, we have $\dim(G_P) \ge 2d$ and by Theorem 9.1.1, this means $\dim(G_P) = \dim(H_d) + 2$. This implies that $\overline{\Omega}_P$ has codimension at least 2 in $\overline{\Omega}(\mathrm{bn}_d)$ by Theorem A.1.9. Proposition 9.2.2 then yields that $\overline{\Omega}_P$ has codimension at least 1 in B.

We will show later that there is a binary vector $\varepsilon \in \{0,1\}^{2d-1}$ and some $t \in \mathbb{C}^\times$ such that $P = \mathrm{bn}_d \circ a$ is linearly equivalent to the polynomial

$$P_{\varepsilon,t} := t \cdot x_1 \cdots x_d + y_1 \cdots y_{d-1} \cdot \left(\sum_{k=1}^{d} \varepsilon_k x_k + \sum_{k=1}^{d-1} \varepsilon_{d+k} y_k \right) \tag{6}$$

We check first that $P_{\varepsilon,t}$ is linearly equivalent to some $\hat{Q}_t$ or to some $Q_{r,s}$. Let r be the number of indices $1 \leq k \leq d$ with $\varepsilon_k = 1$ and let s the number of indices $1 \leq k < d$ with $\varepsilon_{d+k} = 1$. By permutation of the variables we get

$$P \sim P_{\varepsilon,t} \sim t \cdot x_1 \cdots x_d + y_1 \cdots y_{d-1} \cdot \left(\sum_{k=1}^{r} x_k + \sum_{k=1}^{s} y_k \right).$$

If $r = d$ and $s = d - 1$, the above means $P \sim \hat{Q}_t$. If we have $r < d$, then we can scale x_d by t^{-1} to achieve $P \sim Q_{r,s}$. The final case is $s < d - 1$. In this case, choose $\vartheta \in \mathbb{C}$ be so that $\vartheta^d = t$. Scaling y_{d-1} by ϑ^{1-d} and all other variables by ϑ also yields $P \sim Q_{r,s}$.

We will compute the codimension of $\Omega(P_{\varepsilon,t})$ last and first show that $P \sim P_{\varepsilon,t}$. We assume $d = 3$, making the proof easier to read and follow. It generalizes easily to general d. We will show that there are $h \in H_d$ and $g \in \mathrm{GL}(W)$ such that

$$h \circ a \circ g = \begin{pmatrix} t & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & 0 \end{pmatrix} \tag{7}$$

for a certain binary vector ε and $t \in \mathbb{C}^\times$. Then,

$$P_{\varepsilon,t} = \mathrm{bn}_d \circ (h \circ a \circ g) = \mathrm{bn}_d \circ (a \circ g) = P \circ g.$$

To verify (7), we will apply elements of H_d from the left and perform arbitrary column operations to transform a into the right hand side of (7). By column operations, we can achieve

$$a = \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ * & * & * & * & * & 0 \end{pmatrix}$$

with all diagonal entries nonzero. Composing from the right with another element of $\mathrm{GL}(W)$, we may scale all columns to achieve that

$$a = \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & 0 \end{pmatrix}$$

for some binary vector ε. After precomposing a with an element of H_d, we may assume that there are $\tilde{\eta}, \eta \in \mathbb{C}^\times$ with

$$a = \begin{pmatrix} \tilde{\eta} & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{\eta} & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\eta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta & 0 \\ \varepsilon_1\eta & \varepsilon_2\eta & \varepsilon_3\eta & \varepsilon_4\eta & \varepsilon_5\eta & 0 \end{pmatrix}.$$

We now scale every column with η^{-1} and set $\vartheta := \tilde{\eta}\eta^{-1}$, obtaining the form

$$a = \begin{pmatrix} \vartheta & 0 & 0 & 0 & 0 & 0 \\ 0 & \vartheta & 0 & 0 & 0 & 0 \\ 0 & 0 & \vartheta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & 0 \end{pmatrix}$$

Finally, set $t := \vartheta^d$. We multiply by $\mathrm{diag}(\vartheta^{d-1}, \vartheta^{-1}, \dots, \vartheta^{-1}, 1, \dots, 1) \in H_d$ from the left to achieve

$$a = \begin{pmatrix} t & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \varepsilon_5 & 0 \end{pmatrix}. \qquad \blacksquare$$

9.3 The Second Boundary Component

We will use the linear subspace L_0 to construct a concise degeneration Q of bn_d. We choose the linear map $b\colon W \to L_0$ as in (2), i.e., we have $x_i \circ b = x_i$ and $y_i \circ b = \zeta x_i$ for all $1 \le i \le d$, where $\zeta^d = -1$. Consider the linear approximation

$$\mathrm{bn}_d \circ (b + t \cdot \mathrm{id}_W) = (t+1)^d (x_1 \cdots x_d) + \prod_{i=1}^{d} (\zeta x_i + t y_i).$$

The coefficient of t in this expression is the polynomial

$$\tilde{Q} := \sum_{k=1}^{d} \left(y_k \cdot \zeta^{d-1} \prod\nolimits_{\substack{\ell=1 \\ \ell \neq k}}^{d} x_\ell \right) + d \cdot \prod_{\ell=1}^{d} x_\ell.$$

Scaling each x_ℓ by $\sqrt[d]{d^{-1}}$ and each y_k by $\zeta^{d+1} \sqrt[d]{d^{d-1}}$, we get the polynomial

$$Q := \sum_{k=1}^{d} \left(y_k \cdot \prod\nolimits_{\substack{\ell=1 \\ \ell \neq k}}^{d} x_\ell \right) + \prod_{\ell=1}^{d} x_\ell,$$

which satisfies $Q \in \tilde{Q} \circ \mathrm{GL}(W) \subseteq \overline{\Omega}(\mathrm{bn}_d)$.

We now prove that Q is concise and compute the dimension of its stabilizer:

9.3.1 Proposition. The polynomial Q is concise and $\dim(G_Q^\circ) = 2d - 1$.

Furthermore, the identity component G_Q° of G_Q consists of all matrices

$$\left(\begin{smallmatrix} s & 0 \\ u & t \end{smallmatrix}\right) = \left(\begin{array}{ccc|ccc} s_1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & s_d & & & \\ \hline u_1 & & 0 & t_1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & u_d & 0 & & t_d \end{array}\right) \tag{8}$$

such that $t_j = \prod_{i \neq j} s_i^{-1}$ for all $1 \le j \le d$ and $1 = \prod_{i=1}^{d} s_i + \sum_{j=1}^{d} u_j \prod_{i \neq j} s_i$.

Proof. We will compute the Lie algebra of G_Q°. In the process, we will also compute the partial derivatives of Q and note in passing that Q is concise.

Denote by $Y \subseteq \mathrm{GL}(W)$ the closed subvariety of all matrices of the form (8), subject to the listed conditions. We use coordinates $(x_1, \ldots, x_d, y_1, \ldots, y_d)$ on the space $W = \mathbb{C}^d \times \mathbb{C}^d$. In other words, x_i is mapped to $s_i x_i$ and y_j is mapped to $u_j x_j + t_j y_j$. It is a straightforward computation to verify that any element of Y stabilizes Q, so $Y \subseteq G_Q^\circ$. This implies that we have $\dim(G_Q^\circ) \geq \dim(Y) \geq 3d - (d+1) = 2d - 1$ since any of the given $(d+1)$ equations can reduce the dimension by at most one.

We now show that the dimension of the Lie algebra $\mathfrak{Lie}(G_Q)$ is at most $2d-1$. This gives us

$$2d - 1 \leq \dim(G_Q) = \dim(\mathfrak{Lie}(G_Q)) \leq 2d - 1,$$

therefore we have equality. It follows in particular that Y is an irreducible component of G_Q containing the identity, so $Y = G_Q^\circ$.

Recall the action of $\mathfrak{gl}(W)$ on $V = \mathbb{C}[W]_d$ from the proof of Theorem 9.1.1. We again consider the elements of $\mathfrak{gl}(W)$ as block matrices $\left(\begin{smallmatrix} s & v \\ u & t \end{smallmatrix}\right)$ where $s, u, v, t \in \mathbb{C}^{d \times d}$. Such a block matrix is in $\mathfrak{Lie}(G_Q)$ if and only if

$$0 = \sum_{i=1}^{d} \sum_{j=1}^{d} \left(s_{ij} \cdot \left(x_i \frac{\partial Q}{\partial x_j} \right) + u_{ij} \cdot \left(x_i \frac{\partial Q}{\partial y_j} \right) + v_{ij} \cdot \left(y_i \frac{\partial Q}{\partial x_j} \right) + t_{ij} \cdot \left(y_i \frac{\partial Q}{\partial y_j} \right) \right). \tag{9}$$

We set $\mu_j := \frac{x_1 \cdots x_d}{x_j} = \frac{\partial Q}{\partial y_j}$ and similarly $\mu_{jk} := \frac{x_1 \cdots x_d}{x_j x_k}$. Then, we have

$$\frac{\partial Q}{\partial x_j} = \sum_{\substack{k=1 \\ k \neq j}}^{d} \left(y_k \cdot \prod_{\substack{\ell=1 \\ \ell \neq j, \ell \neq k}}^{d} x_\ell \right) + \prod_{\substack{\ell=1 \\ \ell \neq j}}^{d} x_\ell = \sum_{\substack{k=1 \\ k \neq j}}^{d} y_k \mu_{jk} + \mu_j.$$

We note that the partial derivatives of Q are all linearly independent, hence Q is concise – this proves part of the statement. Plugging this into (9) yields

$$\begin{aligned} 0 &= \sum_{i=1}^{d} \sum_{j=1}^{d} \left(u_{ij} x_i \mu_j + t_{ij} y_i \mu_j + \left(\sum_{k \neq j} s_{ij} x_i y_k \mu_{jk} \right) + s_{ij} x_i \mu_j + \left(\sum_{k \neq j} v_{ij} y_i y_k \mu_{jk} \right) + v_{ij} y_i \mu_j \right) \\ &= \sum_{i=1}^{d} \sum_{j=1}^{d} \left((s_{ij} + u_{ij}) x_i \mu_j + (t_{ij} + v_{ij}) y_i \mu_j + \left(\sum_{k \neq j} s_{ij} x_i y_k \mu_{jk} \right) + \left(\sum_{k \neq j} v_{ij} y_i y_k \mu_{jk} \right) \right). \end{aligned} \tag{10}$$

We have summarized some coefficients of certain monomials appearing in the right hand side of (10) in Figure 9.3.1. Coefficient 6 implies that v is the zero matrix and Coefficient 5 implies that $s = \mathrm{diag}(s_1, \ldots, s_d)$ is diagonal. Note that we can pick any $k \notin \{i, j\}$ for Coefficients 5 and 6, and we require the condition $d \geq 3$ for such a k

No.	Monomial	Condition	Coefficient
(1)	$x_i\mu_j$	$i \neq j$	$s_{ij} + u_{ij}$
(2)	$x_j\mu_j$		$\sum_{i=1}^d s_{ii} + \sum_{i=1}^d u_{ii}$
(3)	$y_i\mu_j$	$i \neq j$	$t_{ij} + v_{ij} + s_{ij}$
(4)	$y_k\mu_k$		$t_{kk} + v_{kk} + \sum_{j\neq k} s_{jj}$
(5)	$x_iy_k\mu_{jk}$	$i \neq j,\ k \notin \{i,j\}$	s_{ij}
(6)	$y_iy_k\mu_{jk}$	$j \neq k$	v_{ij}

Figure 9.3.1: Coefficients of certain monomials occuring in (10)

to exist. Coefficient 1 therefore means that $u = \operatorname{diag}(u_1, \ldots, u_d)$ is also diagonal, and Coefficient 2 becomes

$$\sum_{i=1}^{d} u_i + \sum_{i=1}^{d} s_i = 0. \tag{11}$$

Finally, Coefficient 3 implies that $t = \operatorname{diag}(t_1, \ldots, t_d)$ is also diagonal and Coefficient 4 means

$$t_k = -\sum_{\substack{j=1\\ j\neq k}}^{d} s_j.$$

Hence, the only parameters that remain are the diagonal entries of s and u, modulo the relation (11). This proves $\dim(\mathfrak{Lie}(G_Q)) \leq 2d - 1$ as required. ■

9.3.2 Corollary. The orbit closure $\overline{\Omega}(Q)$ is a component of $\partial\Omega(\mathrm{bn}_d)$ which is not contained in $\mathrm{bn}_d \circ \operatorname{End}(W)$.

Proof. Proposition 9.3.1 implies that $\overline{\Omega}(Q)$ is an irreducible closed subvariety of the boundary $\partial\Omega(\mathrm{bn}_d)$. It is not contained in $\mathrm{bn}_d \circ \operatorname{End}(W)$ because Q is concise. Furthermore, this variety has dimension $\dim(\Omega(Q)) = 4d^2 - 2d + 1 = \dim(\Omega(\mathrm{bn}_d)) - 1$, by Corollary 9.1.2 and Theorem A.1.9. ■

9.4 The Indeterminacy Locus

We will ommit bn_d as a subscript in this chapter to ease the notation, i.e., we denote by $\mathcal{A}$ the annihilator of bn_d and by $\hat{\mathcal{A}}$ the subscheme of $\operatorname{End}(W)$ given by the equations $\mathrm{bn}_d \circ a = 0$ in the coordinates a.

We have constructed one concise component of the boundary, but we do not know if there are more, potentially depending on d. Unfortunately, we cannot determine the complete scheme structure of $\hat{\mathcal{A}}^{\mathrm{ss}}$. We know that $\mathcal{A}^{\mathrm{ss}} = H_d \circ \operatorname{End}(W, L_0)^{\mathrm{ss}}$ by (3). We will see that $\mathcal{A}^{\mathrm{ss}}$ has $d!$ irreducible components and each of these components is smooth. However, the scheme $\hat{\mathcal{A}}^{\mathrm{ss}}$ will be singular where these components intersect.

9.4.1 Remark. If $\hat{\mathcal{A}}^{\mathrm{ss}}$ is reduced and its irreducible components intersect transversally, the blowup $\beta\colon \Gamma \to \mathrm{End}(W)^{\mathrm{ss}}$ with center $\hat{\mathcal{A}}^{\mathrm{ss}} = \mathcal{A}^{\mathrm{ss}}$ is a sequence of smooth blowups, one for each component [Li09, Thm. 1.3]. Moreover, in this case the stabilizer of the binomial acts transitively on the irreducible components of $\beta^{-1}(\mathcal{A}^{\mathrm{ss}})$ and by Remark 7.3.12, it follows that $\overline{\Omega}(Q)$ is the only concise component of $\partial\Omega(\mathrm{bn}_d)$. ●

We emphasize that this remark is not to be mistaken as a conjecture – it is quite unclear whether or not $\hat{\mathcal{A}}^{\mathrm{ss}}$ is even reduced at the intersection of its components. That said, we can still show the following:

9.4.2 Theorem. For any point $y \in \mathcal{A}^{\mathrm{ss}}$ that lies outside the intersection of two irreducible components of $\mathcal{A}^{\mathrm{ss}}$, the scheme $\hat{\mathcal{A}}^{\mathrm{ss}}$ is smooth at y. In particular, $\hat{\mathcal{A}}^{\mathrm{ss}}$ is generically smooth and in particular generically reduced.

The remainder of this section is dedicated to the proof of Theorem 9.4.2.

9.4.3. Consider $b = \left(\begin{smallmatrix} \mathbb{I} & 0 \\ \zeta\mathbb{I} & 0 \end{smallmatrix}\right)$, then $\mathcal{A}^{\mathrm{ss}} = (\overline{H_d \circ b \circ \mathrm{GL}(W)})^{\mathrm{ss}}$. Let $T \subseteq \mathrm{SL}_d$ be the group of diagonal matrices with determinant one. We define

$$\begin{aligned} Y_1 := \overline{H_d^\circ \circ b \circ \mathrm{GL}(W)} &= \overline{\left\{ \left(\begin{smallmatrix} s & 0 \\ 0 & t \end{smallmatrix}\right) \left(\begin{smallmatrix} \mathbb{I} & 0 \\ \zeta\mathbb{I} & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right) \;\middle|\; \left(\begin{smallmatrix} s & 0 \\ 0 & t \end{smallmatrix}\right) \in H_d^\circ,\ \left(\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix}\right) \in \mathrm{GL}(W) \right\}} \\ &= \left\{ \left(\begin{smallmatrix} sa \\ \zeta a \end{smallmatrix}\right) \;\middle|\; s \in T \text{ and } a \in \mathbb{C}^{d\times 2d} \right\}. \end{aligned}$$

For a permutation $\pi \in \mathfrak{S}_d$, which we understand as a $d \times d$ permutation matrix, we furthermore define

$$Y_\pi := \left(\begin{smallmatrix} \pi & 0 \\ 0 & \mathbb{I} \end{smallmatrix}\right) \circ Y_1 = \left\{ \left(\begin{smallmatrix} \pi sa \\ \zeta a \end{smallmatrix}\right) \;\middle|\; s \in T \text{ and } a \in \mathbb{C}^{d\times 2d} \right\}.$$ ●

9.4.4 Proposition. The irreducible components of $Y := \overline{H_d \circ b \circ \mathrm{GL}(W)}$ are the Y_π for all permutations $\pi \in \mathfrak{S}_d$ and these components are pairwise distinct.

In particular, Y has $d!$ irreducible components. A point $\left(\begin{smallmatrix} \pi sa \\ \zeta a \end{smallmatrix}\right) \in Y$ is semistable if and only if all rows of a are nonzero.

Proof. Recall the permutation group $K \subseteq \mathrm{GL}(W)$ from Theorem 9.1.1 which permutes the first and last d coordinates among themselves and which also swaps the x and y coordinates simultaneously. Since $H_d = K \cdot H_d^\circ$, we have $Y = K \cdot Y_1$.

We first note that Y_1 is invariant under the permutation switching the x and y coordinates simultaneously: For $s \in T$ and $a \in \mathbb{C}^{d\times 2d}$, set $t := \zeta^2 s^{-1} \in T$ and $c := \zeta^{-1} sa$. Then, $\left(\begin{smallmatrix} \zeta a \\ sa \end{smallmatrix}\right) = \left(\begin{smallmatrix} tc \\ \zeta c \end{smallmatrix}\right)$.

Furthermore, we will see that permutations on the first d coordinates suffice: For $\varrho, \sigma \in \mathfrak{S}_d$ let $\pi := \varrho\sigma^{-1}$ and $b := \sigma a$ to see that

$$\left(\begin{smallmatrix} \varrho & 0 \\ 0 & \sigma \end{smallmatrix}\right) \left(\begin{smallmatrix} sa \\ \zeta a \end{smallmatrix}\right) = \left(\begin{smallmatrix} \varrho sa \\ \sigma\zeta a \end{smallmatrix}\right) = \left(\begin{smallmatrix} \pi sb \\ \zeta b \end{smallmatrix}\right).$$

Assume now that $Y_\pi = Y_\sigma$ for two permutations π and σ. We have $\binom{\pi}{\zeta\mathbb{I}} \in Y_\pi = Y_\sigma$ and hence, there are $a \in \mathbb{C}^{d\times 2d}$ and an $s \in T$ with $\binom{\pi}{\zeta\mathbb{I}} = \binom{\sigma sa}{\zeta a}$. This implies that we have $a = \mathbb{I}$ and hence, $\sigma s = \pi$, so $s = \sigma^{-1}\pi$. However since s is diagonal, this means it must be the identity and therefore $\sigma = \pi$. This proves that the Y_π are the pairwise distinct irreducible components of Y.

The semistability condition follows immediately from the description of the maximal linear subspaces in Section 9.1. ∎

9.4.5 Corollary. A point $\binom{\pi sa}{\zeta a} \in Y_\pi$ lies in the intersection of Y_π with another component Y_σ of Y if and only if there are two rows of $a \in \mathbb{C}^{d\times 2d}$ that are scalar multiples of one another.

Proof. If the i-th and j-th row of a are scalar multiples of one another, let σ be the composition of π with the transposition $\tau := (i,j)$. By assumption, there is a $t \in T$ with $t\tau a = sa$, so we have $\pi sa = \pi\tau t\tau a = \sigma t\tau a = \sigma sa$. Hence, $a \in Y_\pi \cap Y_\sigma$.

Conversely, assume that there is a permutation $\sigma \neq \pi$ with $\binom{\pi sa}{\zeta a} = \binom{\sigma tc}{\zeta c}$ for some $t \in T$ and $c \in \mathbb{C}^{d\times 2d}$. It follows immediately that $c = a$, hence $\pi sa = \sigma ta$. Denoting by a_i the i-th row of a, this means $s_{\pi(i)}a_{\pi(i)} = t_{\sigma(i)}a_{\sigma(i)}$ for all $1 \leq i \leq d$. As $\pi \neq \sigma$, there is an index i with $k := \pi(i) \neq \sigma(i) =: j$, so we have $s_k a_k = t_j a_j$, therefore $a_k = s_k^{-1} t_j a_j$. ∎

The proof of Theorem 9.4.2 is now completed by the following proposition.

9.4.6 Proposition. Let $a \in \mathbb{C}^{d\times 2d}$ be such that no row of a is a scalar multiple of another. Let also $\pi \in \mathfrak{S}_d$ and $u \in T$. Then, $y := \binom{\pi ua}{\zeta a}$ is a smooth point of $\hat{\mathcal{A}}^{\mathrm{ss}}$.

Proof. We first note that $\dim(Y) = \dim(Y_1) = 2d^2 + d - 1$, hence we have to show that the tangent space $\mathrm{T}_y\hat{\mathcal{A}}^{\mathrm{ss}}$ has the same dimension.

As $H_d \times \mathrm{GL}(W)$ and $\mathfrak{S}_d$ act by automorphisms on $\hat{\mathcal{A}}^{\mathrm{ss}}$, the tangent space at y is isomorphic to the tangent space of any element in the $H_d \times \mathrm{GL}(W)$ orbit of y. Hence, we may assume $y = \binom{a}{\zeta a}$.

Note that each row of a is nonzero. After applying some column operation from $\mathrm{GL}(W)$ we may assume that $a_{i1} \neq 0$ and the last d columns of a are zero. Scaling the first column by the inverse of $a_{11}\cdots a_{d1}$ with another column operation, we achieve that $a_{11}\cdots a_{d1} = 1$. We then apply a row scaling operation from H_d to achieve $a_{i1} = 1$ for all $1 \leq i \leq d$. We set $a_i := x_i \circ a = x_1 + \sum_{j=2}^{d} a_{ij}x_j$. Since the a_i are not scalar multiples of one another, we achieve $a_i = x_1 + c_i x_2 + \tilde{a}_i$ with further column operations, where the c_i are all distinct and the $\tilde{a}_i$ are linear forms in $x_3, \ldots, x_d$.

Let $b = \left(\begin{smallmatrix} s & v \\ u & t \end{smallmatrix}\right) \in \mathrm{T}_y \operatorname{End}(W) = \operatorname{End}(W)$ be a tangent vector, where $s, v, u, t \in \mathbb{C}^{d \times d}$. The polynomial $\mathrm{bn}_d \circ (a + bT)$ is equal to

$$\prod_{i=1}^{d} \left(a_i + \sum_{j=1}^{d} s_{ij} x_j T + \sum_{j=1}^{d} v_{ij} y_j T \right) + \prod_{i=1}^{d} \left(\zeta a_i + \sum_{j=1}^{d} u_{ij} x_j T + \sum_{j=1}^{d} t_{ij} y_j T \right).$$

Set $f_i := \prod_{k \neq i} a_k$, then the coefficient of T in this expression is equal to

$$\begin{aligned} &\sum_{i=1}^{d} \sum_{j=1}^{d} s_{ij} f_i x_j + v_{ij} f_i y_j - \zeta^{-1} \sum_{i=1}^{d} \sum_{j=1}^{d} u_{ij} f_i x_j + t_{ij} f_i y_j \\ &\qquad = \sum_{i=1}^{d} \sum_{j=1}^{d} \left(s_{ij} - \zeta^{-1} u_{ij} \right) f_i x_j + \left(v_{ij} - \zeta^{-1} t_{ij} \right) f_i y_j. \end{aligned} \tag{12}$$

By the Jacobian criterion, $b \in \mathrm{T}_y \hat{\mathcal{A}}^{\mathrm{ss}}$ is equivalent to the vanishing of (12) as a polynomial in the x_j and the y_j. We first note that the $f_i \in \mathbb{C}[x_1, \ldots, x_d]_{d-1}$ are linearly independent polynomials: Indeed,

$$f_i = \tilde{f}_i + \prod_{k \neq i} (x_1 + c_k x_2 + \tilde{a}_k) = \prod_{k \neq i} (x_1 + c_k x_2) \tag{13}$$

for certain polynomials $\tilde{f}_i \in \mathbb{C}[x_1, \ldots, x_d]_{d-1}$ whose monomials all contain some variable x_j with $j > 2$. Hence, it is sufficient to show that the $g_i := \prod_{k \neq i} (x_1 + c_k x_2)$ are linearly independent. Assume that $0 = \sum_{i=1}^{d} \lambda_i g_i$ for certain $\lambda_i \in \mathbb{C}$. For any k, we can evaluate this expression at the point $(-c_k, 1)$ and since g_k is the only function from among the g_i that does not vanish, it follows that $\lambda_k = 0$.

Since we have established that the f_i are linearly independent and do not use the y-variables, it follows immediately that the coefficients of $f_i y_j$ in (12) must all vanish, i.e., we have $v = \zeta t$. This constitutes d^2 linear conditions.

If we can show that the polynomials $f_i x_j$ span a subspace of dimension at least $d^2 - d + 1$ inside $\mathbb{C}[x_1, \ldots, x_d]_d$, we are done: It gives us at least $2d^2 - d + 1$ linear conditions on the parameters $\left(\begin{smallmatrix} s & v \\ u & t \end{smallmatrix}\right)$ and hence,

$$\dim \mathrm{T}_y \hat{\mathcal{A}}^{\mathrm{ss}} \leq 4d^2 - (2d^2 - d + 1) = 2d^2 + d - 1 = \dim(Y).$$

Since the other inequality always holds, this implies equality. Now, recall from (13) that $f_i = \tilde{f}_i + g_i$. For $1 \leq k \leq d$, let U_k be the span of all monomials that have combined degree k in $\{x_1, x_2\}$, so we have $\mathbb{C}[x_1, \ldots, x_d]_d = \bigoplus_{k=0}^{d} U_k$. We have $f_i x_j \in \bigoplus_{k=0}^{d-1} U_k$ for all $2 < j \leq d$ and the projection to U_{d-1} is equal to $g_i x_j$. These polynomials span a space of dimension $d^2 - 2d$ because the g_i are linearly independent and do not contain the variables x_j when $2 < j \leq d$.

For $j \in \{1, 2\}$, the projection of $f_i x_j$ to U_d is equal to $g_i x_j$ and it will suffice to show that these polynomials span a space of dimension at least $d + 1$. Write these vectors in coordinates with respect to the basis $(x_1^0x_2^d, x_1^1x_2^{d-1}, x_1^2x_2^{d-2}, \ldots, x_1^dx_2^0)$, then they form the columns of a $(d+1)\times(2d)$ matrix

$$M := \begin{array}{c} \begin{array}{cccccc} g_1x_1 & \cdots & g_dx_1 & g_1x_2 & \cdots & g_dx_2 \end{array} \\ \left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & & & \\ \cline{1-3} & & & & g & \\ & g & & & & \\ \cline{4-6} & & & 0 & \cdots & 0 \end{array}\right) \end{array}$$

with $g \in \mathrm{GL}_d$ because the g_i are linearly independent. We have

$$M \cdot \begin{pmatrix} g^{-1} & 0 \\ 0 & g^{-1} \end{pmatrix} = \left(\begin{array}{ccc|ccc} * & \cdots & * & 1 & & \\ \cline{1-3} 1 & & & & \ddots & \\ & \ddots & & & & 1 \\ \cline{4-6} & & 1 & * & \cdots & * \end{array}\right)$$

which is easily seen to have full rank. ■

As explained in Remark 9.4.1, it would be interesting to know whether $\hat{\mathcal{A}}^{\mathrm{ss}}$ is reduced everywhere:

9.4.7 Question.

(1) Is the scheme $\hat{\mathcal{A}}^{\mathrm{ss}}_{\mathrm{bn}_d}$ Cohen-Macaulay?

(2) Is the scheme $\hat{\mathcal{A}}^{\mathrm{ss}}_{\mathrm{bn}_d}$ reduced?

Remark. Note that (1) implies (2) by Theorem 9.4.2 and the fact that Cohen-Macaulay schemes have no embedded components [GW10, Prop. 14.124]. ○

Part III

Appendix

Appendix A

Algebraic Groups and Representation Theory

Throughout this chapter, we fix an algebraically closed field $\mathbb{C}$ of characteristic zero. We assume knowledge of classical algebraic geometry, see [Har06; Har95] and also [Kra85, Anhang]. We denote the coordinate ring of an affine $\mathbb{C}$-variety X by $\mathbb{C}[X]$. We also use the notation $[n] := \{1, \dots, n\}$.

A.1 Algebraic Semigroups and Groups

If E is a semigroup acting on a set X, we usually denote this action by a dot, so the action of $g \in E$ on $x \in X$ is denoted $g.x$. The set $E_x := \{g \in E \mid g.x = x\}$ is the **stabilizer** of $x \in X$. The **orbit** of x under the action of E is the set $E.x := \{g.x \mid g \in E\}$. If $E.x = \{x\}$, we say that x is E-**invariant**. If $S \subseteq E$ is any subset, we denote by $X^S := \{x \in X \mid \forall s \in S : s.x = x\}$ the set of all S-invariant elements of X. A subset $Y \subseteq X$ is E-**stable** if $E.Y \subseteq Y$. If E is a monoid, we write $\mathbf{1}$ or $\mathbf{1}_E$ for the unique neutral element of E.

A.1.1 Definition. An **algebraic semigroup** is a semigroup $(E, \circ)$ which is also a $\mathbb{C}$-variety such that the composition map

$$\begin{aligned} \mu_E \colon E \times E &\longrightarrow E \\ (g, h) &\longmapsto g \circ h \end{aligned}$$

is a morphism of varieties. A **morphism** of algebraic semigroups is a morphism $\phi : E \to E'$ of varieties between two algebraic semigroups which is also a morphism of semigroups. An **left-E-variety** is a variety X together with a morphism

$$\alpha_X \colon E \times X \longrightarrow X$$

which defines a left E-action on the set X. Similarly, the notion of a **right-E-variety** is defined. An E**-variety** is defined to be a left-E-variety. A **morphism** of E-varieties is a map $\phi\colon X \to Y$ which is a morphism of varieties such that $\phi(g.x) = g.\phi(x)$ for all $g \in E$ and $x \in X$. ●

A.1.2 Definition. An **algebraic group (monoid)** is an algebraic semigroup E which is also a group (monoid). The **unit group** of an algebraic monoid E is defined as the group $G(E) := \{g \in E \mid \mathbf{1} \in (g \circ E) \cap (E \circ g)\}$. ●

A.1.3 Proposition. The unit group of an algebraic monoid is an algebraic group. If G is an algebraic group, then G is a smooth variety and the inversion morphism $\iota_G : G \to G$ mapping $g \mapsto g^{-1}$ is a morphism of varieties.

Proof. We first show that G is smooth. The set of regular points of G is open and dense, therefore not empty. Let's assume that G is regular at g. Given any other point $h \in G$, since the multiplication with the group element hg^{-1} is an automorphism of G, it follows that $h = hg^{-1}g$ is also a nonsingular point.

Let E be an algebraic monoid and $G = G(E)$. The set $\Gamma := \mu_E^{-1}(\mathbf{1}) \subseteq E \times E$ is a closed subvariety because it is the preimage of a closed point. For $i \in \{1, 2\}$ we denote by $\mathrm{pr}_i \colon E \times E \to E$ the projections. Then, $G = \mathrm{pr}_1(\Gamma) \cap \mathrm{pr}_2(\Gamma)$ is constructible by [TY05, 15.4.3]. This means that G is locally closed. Hence, it contains an open subset U of its closure $\overline{G}$ in E. Because G acts transitively on itself, we see that $G = \bigcup_{g \in G} g.U$ is open in $\overline{G}$, therefore a variety and consequently, an algebraic group.

Since any algebraic group G is the unit group of itself, we can maintain the above notation for the rest of the proof. We now show that ι_G is a morphism of varieties. Note that Γ is the graph of the map ι_G. Let $\pi_i := \mathrm{pr}_i|_\Gamma$ be the restrictions of the projection morphisms to Γ. Each of the π_i is a bijective map from a variety to a smooth variety, so by [TY05, Corollary 17.4.8] they are both isomorphisms of varieties. This implies that $\iota_G = \pi_2^{-1} \circ \pi_1$ is a morphism. ∎

A.1.4 Example. Let $W \cong \mathbb{C}^n$ and $E = \mathrm{End}(W) \cong \mathbb{C}^{n \times n}$. E is an algebraic monoid and the unit group of E is the general linear group $G = \mathrm{GL}(W)$ of invertible linear maps $W \to W$. By choosing a basis of W, we can identify G with the group GL_n of invertible complex $n \times n$ matrices. We consider the set $V = \mathbb{C}[W]_d$ of homogeneous d-forms on W. It is a finite-dimensional $\mathbb{C}$-vector space, hence a variety. We define the (right) action $\alpha_V \colon V \times E \to V$ via $\alpha_V(P, a) := P \circ a$. ●

A.1.5 Remark. Let G be an algebraic group. If X is an affine G-variety, we have an induced G-action on the coordinate ring $\mathbb{C}[X]$ by $g.\varphi := (g^{-1})^*(\varphi) = \varphi \circ g^{-1}$, where we understand $g^{-1} \colon X \to X$ as a morphism of varieties. Indeed, for $g, h \in G$ and $\varphi \in \mathbb{C}[X]$, $g.h.\varphi = (g^{-1})^*((h^{-1})^*(\varphi)) = (h^{-1}g^{-1})^*(\varphi) = ((gh)^{-1})^*(\varphi) = gh.\varphi$.

The inverse is required to obtain a *left* action rather than a right action. ●

The following result will be important to reduce to the connected case:

A.1.6 Proposition. Let G be an algebraic monoid. There is a unique irreducible component $G^\circ \subseteq G$ containing $\mathbf{1}$. G° is a closed submonoid of G.

If G is an algebraic group, then G° is a normal subgroup and the irreducible components of G are the cosets hG° for $h \in G$. In particular, the quotient group G/G° is finite.

Proof. Let G° be a connected component of G containing $\mathbf{1}$ and let X be any irreducible component of G. Then XG° is the image of the restriction of μ_G to $X \times G^\circ$. This means that XG° is the image of an irreducible variety under a regular map, therefore $\overline{XG^\circ}$ is an irreducible, closed subvariety of G. Since $\mathbf{1} \in G^\circ$, we have $X \subseteq XG^\circ \subseteq \overline{XG^\circ}$. X is irreducible, so $X = \overline{XG^\circ}$. This implies $X = XG^\circ$ and similarly, we obtain $X = G^\circ X$. In particular, $G^\circ = G^\circ G^\circ$, so G° is a closed submonoid. If $\mathbf{1}$ was also contained in X, then we could similarly conclude that $G^\circ X = XG^\circ = G^\circ$, so we have $X = G^\circ$.

Assume now that G is a group. If X is an irreducible component, then for any $h \in X$, we know that $h^{-1}X$ is the image of an irreducible set under the automorphism h, so it is an irreducible component. Since $\mathbf{1} \in h^{-1}X$, we know that $h^{-1}X = G^\circ$, hence $X = hG^\circ$. For any $h \in G$, similarly observe that $hG^\circ h^{-1}$ is a component containing $\mathbf{1}$, so $hG^\circ h^{-1} = G^\circ$. This proves that G° is a normal subgroup. Since its cosets are the (finitely many) irreducible components of X, this proves the statement. ∎

A.1.1 Quotients by Algebraic Groups

Let X be a variety on which an algebraic group G acts. Then, a **categorical quotient** is a morphism $\pi\colon X \to Y$ such that

(C1) π is constant on G-orbits.

(C2) If $\psi\colon X \to Z$ is any morphism of varieties which is constant on G-orbits, then there exists a unique $\bar{\psi}\colon Y \to Z$ with $\bar{\psi} \circ \pi = \psi$.

If a categorical quotient exists, it is unique up to unique isomorphism and we denote it by $Y = X /\!\!/ G$.

However, categorical quotients can lack many properties one would expect from the quotient by a group action. For example, there are cases when the projection morphism is not surjective. A much stronger notion is the one of a **good quotient**, defined to be a morphism $\pi\colon X \to Y$ with the following properties:

(G1) The morphism π is G-invariant and surjective.

(G2) The comorphism of π induces an isomorphism $\mathcal{O}_Y \cong (\pi_* \mathcal{O}_X)^G$.

(G3) For a closed and G-invariant subset $Z \subseteq X$, the image $\pi(Z)$ is closed in Y.

(G4) Two closed, G-invariant subsets $Z_1, Z_2 \subseteq X$ are disjoint if and only if $\pi(Z_1)$ and $\pi(Z_2)$ are disjoint.

A good quotient is called a **geometric quotient** if the fibers of π are precisely the G-orbits of X.

A.1.7 Remark. See [TY05, 25.2] for some remarks on Property (G2). The gist is that for any open subset $U \subseteq Y$, the ring $(\pi_* \mathcal{O}_X)(U) = \mathcal{O}_X(\pi^{-1}(U))$ carries a G-module structure, so we can consider the corresponding subring of G-invariant functions $\mathcal{O}_X(\pi^{-1}(U))^G$. This defines a subsheaf of rings $(\pi_*\mathcal{O}_X)^G \subseteq (\pi_*\mathcal{O}_X)$. ●

By [New78, §4, Prop. 3.11], any good quotient is also a categorical quotient. If the quotient is geometric, we denote it by X/G.

A.1.8 Theorem. Let G be an affine, reductive algebraic group acting on an affine variety X. The ring $\mathbb{C}[X]^G$ is a finitely generated $\mathbb{C}$-algebra and $Y = \operatorname{Spec}(\mathbb{C}[X]^G)$ together with the surjective morphism $\pi\colon X \to Y$ induced by $\mathbb{C}[X]^G \subseteq \mathbb{C}[X]$ defines a good quotient of X by G. Furthermore:

(1) If G is finite and X irreducible, then this quotient is geometric.

(2) For all $x \in X$, the fiber $\pi^{-1}(\pi(x))$ contains a unique closed G-orbit Ω. Furthermore, we have $\pi^{-1}(\pi(x)) = \{ y \in X \mid \Omega \subseteq \overline{G.y} \}$.

Proof. See [TY05, 27.5] and [TY05, 25.5.2]. ■

If a group G acts on a set X, one can identify the orbit $G.x$ of any element $x \in X$ with the set of cosets G/G_x where $G_x = \{ g \in G \mid g.x = x \}$ is the stabilizer of x. It turns out that under certain assumptions, this remains valid and compatible when X is a G-variety.

A.1.9 Theorem ([TY05, 21.4]). Let G be an affine algebraic group and X a G-variety. For every point $x \in X$, we have:

(1) The orbit $G.x$ is a smooth subvariety of X which is open in its closure $\overline{G.x}$.

(2) The stabilizer G_x is a closed subgroup of G and every component of $G.x$ has dimension $\dim(G) - \dim(G_x)$.

(3) $\overline{G.x}$ is the union of $G.x$ and orbits of strictly smaller dimension.

(4) If X is affine and G_x is reductive, the right action of G_x on G by multiplication admits the good quotient $G.x \cong G/\!\!/G_x$ and in particular $G.x$ is an affine variety with coordinate ring $\mathbb{C}[G.x] \cong \mathbb{C}[G]^{G_x}$. ■

A.1.2 Nullcone and Projective Quotients

If an algebraic group G acts on a vector space V, the set $\mathcal{N} := \{ v \in V \mid 0 \in \overline{G.v} \}$ is called the **nullcone** of this action. Especially when G is reductive, the geometry of $\mathcal{N}$ carries a lot of information about the quotient, see [Kra85]. We will require the following observations:

A.1.10 Lemma. Let G be a reductive affine algebraic group acting on a complex vector space V. Let $\pi\colon V \twoheadrightarrow V/\!\!/G$ be the quotient and $\mathcal{N} \subseteq V$ the nullcone. Then,

- $\mathcal{N} = \pi^{-1}(\pi(0))$, in particular $\mathcal{N}$ is a closed subset of V.
- $\mathcal{N} = \{v \in V \mid 0 \in \overline{G^\circ.v}\}$ where G° is the connected identity component of G.
- $\mathcal{N}$ is the vanishing set of all homogeneous, G-invariant functions in $\mathbb{C}[V]$.

Proof. Since the origin 0 is a closed G-orbit, we have $0 \in \overline{G.v}$ if and only if $\pi(v) = \pi(0)$ according to Theorem A.1.8.

For the second item, note that $G = H \cdot G^\circ$ where H is a finite set, according to Proposition A.1.6. Since $\overline{G.v} = \overline{H.G^\circ.v} = H.\overline{G^\circ.v}$, we can see that $0 \in \overline{G.v}$ if and only if $0 \in \overline{G^\circ.v}$.

Finally, the defining ideal of $\pi(0)$ in $\mathbb{C}[V]^G$ is equal to the maximal ideal generated by all homogeneous elements of positive degree, and since π corresponds to the inclusion $\mathbb{C}[V]^G \subseteq \mathbb{C}[V]$, the set $\mathcal{N}$ is the vanishing set of the same elements, considered as functions on V. ∎

If V is a G-module, the projective space $\mathbb{P}(V)$ is a G-variety in a natural way. A **linearized** projective G-variety is a projective variety X with a fixed embedding as a closed, G-invariant subset $X \subseteq \mathbb{P}(V)$ for some G-module V. We write $\check{X} \subseteq V$ for the affine cone over X, i.e., $\check{X}$ is an affine variety which is closed under scalar multiplication and $X = \mathbb{P}(\check{X})$.

If X is a linearized projective G-invariant closed cone, then $\mathbb{C}[\check{X}]$ is a graded algebra. It is the projective coordinate ring of X. The invariant algebra $\mathbb{C}[\check{X}]^G$ inherits this grading and in particular, one can consider the projective variety $Y := \operatorname{Proj}(\mathbb{C}[\check{X}]^G)$ and the induced rational map $\pi\colon X \dashrightarrow Y$. If we denote by $[x] \in X$ the projective class of a point $x \in \check{X}$, then $\pi([x]) = [\check{\pi}(x)]$ where $\check{\pi}\colon \check{X} \twoheadrightarrow \check{X}/\!\!/G$ is the affine quotient from Theorem A.1.8. This rational map is undefined when $\check{\pi}(x) = 0$, which by Lemma A.1.10 means that $\mathbb{P}\mathcal{N}$ is the indeterminacy locus of π. Hence, $X^{\mathrm{ss}} := X \setminus \mathbb{P}\mathcal{N}$ is the open set where π is defined. It is called the set of **semistable** points of X, with respect to the (induced) action of G on X. One also calls $\check{X}^{\mathrm{ss}} := \check{X} \setminus \mathcal{N}$ the set of semistable points of $\check{X}$. Since $\mathcal{N}$ is a cone, we have $X^{\mathrm{ss}} = (\mathbb{P}\check{X})^{\mathrm{ss}} = \mathbb{P}(\check{X}^{\mathrm{ss}})$ and we sometimes omit the brackets. The restriction $\pi\colon X^{\mathrm{ss}} \to Y$ is a morphism.

A.1.11 Proposition. Let X be a linearized projective G-variety and S its projective coordinate ring. The projective variety $Y := \operatorname{Proj}(S^G)$ together with the morphism $\pi\colon X^{\mathrm{ss}} \to Y$ induced by $S^G \subseteq S$ defines a good quotient $Y = X^{\mathrm{ss}}/\!\!/G$.

Using Theorem A.1.8, the proof is straightforward. We omit it for brevity and refer to [Dol03, Thm. 8.1, Prop. 8.1] for a modern treatment and [New78, §4, Thm. 3.14] for a more classical version. We note the following amplification of Property (C2):

A.1.12 Proposition. Let X be a linearized projective G-variety and let $\varpi\colon X \dashrightarrow Y$ be a G-invariant rational map between projective varieties. Then, there exists a rational map $\phi\colon X^{\mathrm{ss}} /\!\!/ G \dashrightarrow Z$ such that the following diagram commutes:

$$\begin{tikzcd} X^{\mathrm{ss}} \arrow[rr, dashed, "\varpi"] \arrow[dr, two heads, "\pi"'] & & Y \\ & X^{\mathrm{ss}} /\!\!/ G \arrow[ur, dashed, "\phi"'] & \end{tikzcd}$$

Here, π is the quotient morphism.

Proof. Fix some embedding of Y in a projective space. Let S and R be the projective coordinate rings of X and Y, respectively. The rational map ϖ corresponds to a homomorphism $\varpi^\sharp\colon R \to S$ of graded $\mathbb{C}$-algebras and the assumption that ϖ is G-invariant means that the image of $\varpi^\sharp$ is contained in the invariant ring S^G. Since S^G is the projective coordinate ring of $X^{\mathrm{ss}} /\!\!/ G$, this homomorphism $R \to S^G$ induces a rational map $\phi\colon X^{\mathrm{ss}} /\!\!/ G \dashrightarrow Y$. The inclusion $S^G \hookrightarrow S$ of graded rings corresponds to the rational map $\tilde{\pi}\colon X \dashrightarrow X^{\mathrm{ss}} /\!\!/ G$ which restricts to π, so we have $\phi \circ \tilde{\pi} = \varpi$ as rational maps. Restricting to X^{ss} yields the result. ∎

A.2 Representation Theory of Reductive Groups

Let G be an affine algebraic group throughout this section. A **Borel subgroup** of G is a maximal connected solvable subgroup B of G. It follows from the definition that a Borel subgroup of G is closed, because if $H \subseteq G$ is connected and solvable, then so is its closure. By [Hum98, 21.3], G/B is a projective variety and all Borel subgroups of G are conjugate. We fix one Borel B of G.

The only connected, algebraic $\mathbb{C}$-groups of dimension one are the **multiplicative group** $\mathbb{G}_{\mathrm{m}} := \mathbb{G}_{\mathrm{m}}(\mathbb{C}) := (\mathbb{C}^\times, \cdot)$ and the **additive group** $\mathbb{G}_{\mathrm{a}} := \mathbb{G}_{\mathrm{a}}(\mathbb{C}) := (\mathbb{C}, +)$, see [TY05, 22.6.2]. A product of multiplicative groups $\mathbb{G}_{\mathrm{m}}^r$ is called a **torus**.

G contains a unique largest normal solvable subgroup [TY05, 27.1.1], which is automatically closed. Its identity component is then the largest connected normal solvable subgroup of G and is called the **radical** of G, denoted $\mathrm{R}(G)$. By [TY05, 27.1.2], the set of unipotent elements of $\mathrm{R}(G)$ is the largest connected normal unipotent subgroup of G. It is denoted by $\mathrm{R_u}(G)$ and we call it the **unipotent radical** of G. An affine algebraic group G is called **semisimple** (resp. **reductive**) if $\mathrm{R}(G)$ (resp. $\mathrm{R_u}(G)$) is trivial. Note that we follow [TY05, 27.2] here and do not require G to be connected.

However, an algebraic group G has a unique, connected, normal subgroup G° containing the neutral element [TY05, 21.1.5 & 21.1.6] and by definition, G is semisimple (resp. reductive) if and only if G° has this property.

For example, each torus is reductive. Furthermore, GL_n is reductive, but not semisimple. However, the subgroup $\mathrm{SL}_n \subseteq \mathrm{GL}_n$ of matrices with determinant 1 is semisimple.

By [Hum98, 21.3], the maximal tori and the maximal unipotent subgroups of G are those of the Borel subgroups of G, and all of them are also conjugate. We fix a maximal torus $T \subseteq B$ and denote by $U = \mathrm{R_u}(B)$ the unipotent radical of B. We have $B = U \rtimes T$ by [Hum98, 19.3].

A.2.1 Example. The affine space $\mathbb{C}^{n\times n}$ of all $n \times n$ matrices with entries in $\mathbb{C}$ has the coordinate ring $\mathbb{C}\left[x_{ij} \,\middle|\, i,j \in [n]\right]$. The group $G := \mathrm{GL}_n$ of invertible $n \times n$ matrices is the set of matrices where $\det_n$ does not vanish and therefore a Zariski-open subset of $\mathbb{C}^{n\times n}$. Hence, G is an affine algebraic group with coordinate ring $\mathbb{C}[x_{ij}, \det_n^{-1}]$. Denote by $S := \mathrm{SL}_n$ the set of matrices with determinant 1, i.e., the vanishing set of $\det_n -1$. It is a closed, affine algebraic subgroup of G.

Denote by $\mathbb{I}_n$ be the unit matrix, then it is well-known that the center of $G = \mathrm{GL}_n$ is equal to $Z := \mathbb{C}^\times \cdot \mathbb{I}_n$. By [Lan02, Part 3, Chapter XIII, 8 & 9], the projective linear group $\mathrm{PGL}_n = G/Z = S/(S \cap Z)$ is simple. Thus, if $R \trianglelefteq G$ is solvable, then R/Z is also solvable and can only be trivial or equal to PGL_n. However, PGL_n is not solvable because S is perfect (i.e., $[S,S] = S$) and therefore not solvable. Thus, R/Z is trivial, which means $R \subseteq Z$. Since Z is connected, normal and solvable, we must have $\mathrm{R}(G) = Z$. The only unipotent central matrix is the unit matrix, so $\mathrm{R_u}(G)$ is trivial and we have verified that G is reductive.

A maximal Borel subgroup B of G is given by the upper triangular matrices of nonvanishing determinant and its unipotent radical U is the subgroup of those elements where all diagonal entries are equal to 1. The maximal torus T corresponding to this choice are the invertible diagonal matrices. ●

A.2.1 Representations

Let G be an algebraic group. A **representation** of G is a homomorphism of algebraic groups $\varrho\colon G \to \mathrm{GL}(\mathbb{V}(\varrho))$, where $\mathbb{V}(\varrho)$ is some *finite-dimensional* $\mathbb{C}$-vector space. The space $\mathbb{V}(\varrho)$ becomes a G-variety via the induced action. We also write $\mathbb{V}_G(\varrho)$ instead of $\mathbb{V}(\varrho)$ if we want to put emphasis on the group. The **degree** of ϱ is defined to be $\dim(\mathbb{V}(\varrho))$.

Conversely, if V is some $\mathbb{C}$-vector space which is also a G-variety, we say that V is a G-**module**. We remark that the G-modules are in bijection with the representations of G by associating to a representation $\varrho\colon G \to \mathrm{GL}(V)$ the action $\alpha\colon G \times V \to V$ where $\alpha(g,v) := \varrho(g)(v)$.

A **homomorphism** of G-representations is a $\mathbb{C}$-linear map $\phi\colon \mathbb{V}_G(\varrho_1) \to \mathbb{V}_G(\varrho_2)$ such that $\varrho_2(g) \circ \phi = \phi \circ \varrho_1(g)$ for all $g \in G$. Equivalently, a homomorphism of G-modules is a $\mathbb{C}$-linear map $\phi\colon V_1 \to V_2$ where $\phi(g.x) = g.\phi(x)$ for all $g \in G$.

We say that ϱ_1 is a **subrepresentation** of ϱ_2 if there is an injective homomorphism of G-representations $\mathbb{V}_G(\varrho_1) \hookrightarrow \mathbb{V}_G(\varrho_2)$. Equivalently, a submodule of a G-module V_1 is a linear subspace $V_1 \subseteq V_2$ such that V_1 is stable under the action of G. A representation ϱ is **irreducible** if ϱ has no nontrivial subrepresentations. Equivalently a G-module V is irreducible if it has no G-invariant linear subspaces other than V and $\{0\}$. Denote by $\mathrm{Rep}(G)$ the set of all equivalence classes of G-representations and by $\mathrm{Irr}(G) \subseteq \mathrm{Rep}(G)$ the subset of all irreducible representations. If $H \subseteq G$ is a subgroup, we have a canonical restriction map $\mathrm{Rep}(G) \to \mathrm{Rep}(H)$, $\varrho \mapsto \varrho|_H$.

The arguably most important lemma of representation theory is the following:

A.2.2 Lemma (Schur's Lemma). Let G be an algebraic group and $\varrho_1, \varrho_2 \in \mathrm{Irr}(G)$. Every nonzero homomorphism $\phi\colon \mathbb{V}(\varrho_1) \to \mathbb{V}(\varrho_2)$ of G-representations is an isomorphism.

Proof. Note that for $v \in \ker(\phi)$, we have $\phi(g.v) = g.\phi(v) = g.0 = 0$, hence $\ker(\phi)$ is a G-invariant subspace of $\mathbb{V}(\varrho_1)$. Since ϕ is nonzero, we must have $\ker(\phi) = \{0\}$. ∎

For two representations $\varrho_1, \varrho_2 \in \mathrm{Rep}(G)$ with $V_i := \mathbb{V}(\varrho_i)$, we define the representation $\varrho_1 \oplus \varrho_2\colon G \to \mathrm{GL}(V_1 \oplus V_2)$ via $g \mapsto \varrho_1(g) \oplus \varrho_2(g)$. For $\varrho \in \mathrm{Rep}(G)$, let $\varrho^*\colon G \to \mathrm{GL}(\mathbb{V}_G(\varrho)^*)$ be the representation defined by $\varrho^*(g) := (\varrho(g)^*)^{-1}$, i.e., the group homomorphism corresponding to the action

$$\begin{aligned} G \times \mathbb{V}_G(\varrho)^* &\longrightarrow \mathbb{V}_G(\varrho)^* \\ (g,\varphi) &\longmapsto \varphi \circ \varrho(g)^{-1} \end{aligned} \tag{1}$$

A.2.2 Characters, Roots, Weights

Assume henceforth that G is reductive, we have chosen a Borel $B \subseteq G$ and a maximal torus $\mathbb{G}_\mathrm{m}^r \cong T \subseteq B$.

A multiplicative (resp. additive) **character** of an algebraic group H is a homomorphism of algebraic groups $H \to \mathbb{G}_\mathrm{m}$ (resp. $H \to \mathbb{G}_\mathrm{a}$). Conversely, a multiplicative (resp. additive) **one-parameter-subgroup** (also called **1-psg** for short) is a homomorphism of algebraic groups $\mathbb{G}_\mathrm{m} \to H$ (resp. $\mathbb{G}_\mathrm{a} \to H$). When we say character or 1-psg, we refer to the multiplicative version. The set of characters $\mathbb{X}(H)$ and the set of 1-psgs $\check{\mathbb{X}}(H)$ become abelian groups under pointwise multiplication. There are canonical isomorphisms

$$\begin{aligned} \mathbb{Z}^r &\longrightarrow \mathbb{X}(\mathbb{G}_{\mathrm{m}}^r) \\ \lambda &\longmapsto \chi_\lambda \end{aligned} \qquad\qquad \begin{aligned} \mathbb{Z}^r &\longrightarrow \check{\mathbb{X}}(\mathbb{G}_{\mathrm{m}}^r) \\ \gamma &\longmapsto \check{\chi}_\gamma \end{aligned} \tag{2}$$

where χ_λ and $\check{\chi}_\gamma$ are defined as

$$\begin{aligned} \chi_\lambda\colon \mathbb{G}_{\mathrm{m}}^r &\longrightarrow \mathbb{G}_{\mathrm{m}} \\ t &\longmapsto t^\lambda := t_1^{\lambda_1}\cdots t_r^{\lambda_r} \end{aligned} \qquad\qquad \begin{aligned} \check{\chi}_\gamma\colon \mathbb{G}_{\mathrm{m}} &\longrightarrow \mathbb{G}_{\mathrm{m}}^r \\ x &\longmapsto t^\gamma := (t^{\gamma_1},\ldots,t^{\gamma_r}). \end{aligned}$$

For $(\gamma,\lambda) \in \check{\mathbb{X}}(H) \times \mathbb{X}(H)$ we define $\langle\gamma,\lambda\rangle \in \mathbb{Z}$ to be the integer corresponding to $\lambda \circ \gamma \in \mathbb{X}(\mathbb{G}_{\mathrm{m}}) \cong \mathbb{Z}$. In other words, $g^{\langle\gamma,\lambda\rangle} = (g^\gamma)^\lambda$ for all $t \in \mathbb{C}^\times$. It is well-known [TY05, 22.5.2] that $\mathbb{X}(T) \cong \mathrm{Irr}(T)$ in the sense that any $\varrho \in \mathrm{Rep}(T)$ decomposes as a direct sum of **weight spaces**

$$\mathbb{V}(\varrho)_\alpha = \{v \in \mathbb{V}(\varrho) \mid \forall t \in T\colon t.v = t^\alpha v\}.$$

If $\varrho \in \mathrm{Rep}(G)$, we can also regard $\mathbb{V}(\varrho)$ as a T-module and denote by $\mathbb{V}(\varrho)_\alpha$ the corresponding weight space. We call $\Lambda(G) := \mathbb{X}(T)$ the **weight lattice** of G.

To avoid an introduction to the theory of Lie algebras ([Hum98, Chapter III] or [TY05, 23]), we deviate from [Hum98] in defining $\alpha \in \Lambda(G)$ to be a **root** of G with respect to T to be a if there exists an additive 1-psg $\varepsilon_\alpha : \mathbb{G}_{\mathrm{a}} \to G$ such that

$$\forall t \in T\colon \forall x \in \mathbb{C}\colon t \cdot \varepsilon_\alpha(x) \cdot t^{-1} = \varepsilon_\alpha(t^\alpha x).$$

This definition of root is equivalent to the one given in [Hum98, 16.4]: Indeed, one direction of this equivalence is stated in [Hum98, 26.3] and the proof actually verifies the other direction as well.

Denote by $\Phi := \Phi(G,T) \subseteq \Lambda(G)$ the set of roots of G with respect to T. See [Hum98, 27] for a proof that Φ is a so-called *abstract root system* and [Hum80, III.9.2] or [TY05, 18] for general facts about abstract root systems. For $\alpha \in \Phi$, set $U_\alpha := \varepsilon_\alpha(\mathbb{G}_{\mathrm{a}})$. We call $\Phi^+ := \{\alpha \in \Phi \mid U_\alpha \subseteq B\}$ the **positive roots** of G with respect to B.

A.2.3 Example. We continue Example A.2.1 and keep the notations already established. We identify $\Lambda(G) = \mathbb{X}(T)$ with $\mathbb{Z}^n$ in the above way. For $i,j \in [n]$ and $i \neq j$, denote by $\alpha_{ij} \in \mathbb{Z}^n$ the vector with the value 1 in position i, the value -1 in position j and 0 elsewhere. We claim that these weights are the roots of G with respect to T.

For $i,j \in [n]$, we define $E_{ij} \in \mathbb{C}^{n\times n}$ to be the matrix with:

$$x_{rs}(E_{ij}) = \begin{cases} 1 & ; \quad (r,s) = (i,j) \\ 0 & ; \quad (r,s) \neq (i,j) \end{cases}$$

With these notations, we define the map

$$\begin{aligned}\varepsilon_{ij}\colon \mathbb{G}_{\mathrm{a}} = \mathbb{C} &\longrightarrow G\\ x &\longmapsto \mathbb{I}_n + x\cdot E_{ij}\end{aligned}$$

It is elementary to check that this is a morphism of algebraic groups and furthermore, for any $x \in \mathbb{C}$ and any diagonal matrix $t = (t_1, \ldots, t_n) \in T$,

$$\begin{aligned} t\cdot\varepsilon_{ij}(x)\cdot t^{-1} &= t\cdot(\mathbb{I}_n + x\cdot E_{ij})\cdot t^{-1} = t\mathbb{I}_n t^{-1} + x\cdot tE_{ij}t^{-1}\\ &= \mathbb{I}_n + x\cdot \tfrac{t_i}{t_j}\cdot E_{ij} = \varepsilon_{ij}(t^{\alpha_{ij}}x).\end{aligned}$$

We therefore know that the α_{ij} are certainly roots. With more prelude and the theory of Lie algebras, one can see rather directly that these are, in fact, *all* the roots. See [Kra85, Bemerkung 4, III.1.4] or [GW09, Theorem 2.4.1] for a quite elementary treatment.

The group $U_{ij} := U_{\alpha_{ij}}$ is now the image of ε_{ij} and from the definition we can see that $U_{ij} = \{\mathbb{I}_n + t\cdot E_{ij} \mid t \in \mathbb{C}\}$ consists of matrices with 1 on the main diagonal and zeros everywhere else except at position (i, j). Since we chose B to be the group of upper triangular matrices, we can see that $\alpha_{ij} \in \Phi^+$ if and only if $i < j$. ●

It can be shown that there exists a (unique) subset $\Delta \subseteq \Phi$ of linearly independent weights with the property that for all $\alpha \in \Phi^+$, there are nonnegative integers c_δ such that $\alpha = \sum_{\delta\in\Delta} c_\delta \delta$. See [Hum80, III.10.1] or [TY05, 18.7.4] for a proof of this result. A root belonging to Δ is called a **simple root** and $|\Delta|$ is called the **rank** of G.

For every root $\alpha \in \Lambda(G) = \mathbb{X}(T)$, there is a unique dual root $\check{\alpha} \in \check{\mathbb{X}}(T)$ satisfying the two conditions

- $\langle \check{\alpha}, \alpha\rangle = 2$ and
- $\forall \beta \in \Phi\colon \beta - \langle\check{\alpha}, \beta\rangle\cdot\alpha \in \Phi$.

This is part of the root system axioms, see [Hum80, III.9.2] or [TY05, 18.2.1]. Note that the map $\beta \mapsto \beta - \langle\check{\alpha}, \beta\rangle\cdot\alpha$ is referred to as σ_α in [Hum80], as $s_{\alpha,\check{\alpha}}$ in [TY05] and as τ_α In [Hum98]. It is called the **reflection** relative to α. It maps α to $-\alpha$ and leaves a certain hyperplane H_α invariant. We then call

$$\Lambda^+(G) := \{\lambda \in \Lambda(G) \mid \forall\alpha \in \Phi^+\colon \langle\check{\alpha}, \lambda\rangle \geq 0\}$$

the **dominant weights** of G, the notation being justified in that they lie on the side of H_α which we have marked as "positive" by our choice of Φ^+, which in turn is based on the choice of B. See [TY05, 18.11.7] for a proof that $\Lambda^+(G)$ is a finitely generated semigroup.

A.2.4 Example. We continue Example A.2.3. For the set Δ, we choose in this case the roots $\alpha_i := \alpha_{i,(i+1)}$ for $i \in [n-1]$. There are $n-1$ of these roots, they are linearly independent and for $i < j$, we have

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}.$$

The rank of GL_n is therefore $n-1$.

Given any root $\alpha \in \mathbb{Z}^n \cong \mathbb{X}(T)$, we define its dual to be the 1-psg corresponding to the same tuple via (2). With this definition, $\langle \check{\alpha}, \alpha \rangle$ corresponds simply to the ordinary scalar product $\mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ given by $(\alpha, \beta) \mapsto \alpha^{\mathrm{t}}\beta$. We identify α and $\check{\alpha}$. Clearly, $\langle \alpha_{ij}, \alpha_{ij} \rangle = \alpha_{ij}{}^{\mathrm{t}}\alpha_{ij} = 2$.

Let $\lambda \in \mathbb{Z}^n$. For $i < j$, we have $\lambda_i \geq \lambda_j$ if and only if $\langle \alpha_{ij}, \lambda \rangle = \lambda_i - \lambda_j \geq 0$. Hence, the dominant weights of GL_n are precisely the weakly descending vectors of integer numbers – such a vector is called a **generalized partition**.

We remark that in this case, it is easy to see that the semigroup $\Lambda^+(\mathrm{GL}_n)$ is finitely generated: The vectors

$$\begin{aligned} \omega_1 &:= (1, 0, 0, \ldots, 0), \\ \omega_2 &:= (1, 1, 0, \ldots, 0), \\ &\vdots \\ \omega_n &:= (1, 1, 1, \ldots, 1) \end{aligned} \tag{3}$$

form a generating set together with $-\omega_n$. ●

We denote by $\underline{\Lambda}^{\mathrm{r}}(G)$ the $\mathbb{N}$-span of Φ^+. The group $\Lambda^{\mathrm{r}}(G)$ generated by $\underline{\Lambda}^{\mathrm{r}}(G)$ is equal to the $\mathbb{Z}$-span of all roots and is sometimes called the **root lattice** of G.

For two characters $\lambda, \mu \in \mathbb{X}(T)$ we write $\lambda \trianglerighteq \mu$ if $\lambda - \mu \in \underline{\Lambda}^{\mathrm{r}}(G)$.

A.2.5 Example. Still continuing Example A.2.4, we remark finally that the root lattice of $G = \mathrm{GL}_n$ consists of all $\lambda \in \mathbb{Z}^n$ with the property that the sum of all entries vanishes, i.e., $|\lambda| := \lambda_1 + \cdots + \lambda_n = 0$. Furthermore,

$$\underline{\Lambda}^{\mathrm{r}}(G) = \{\lambda \in \mathbb{Z}^n \mid \forall k \in [n] \colon \lambda_1 + \cdots + \lambda_k \geq 0, |\lambda| = 0\}.$$

Indeed, if $\lambda = \sum_{k=1}^{n-1} c_k \alpha_k$ is a sum of simple roots with $c_k \in \mathbb{Z}$, then $\lambda_k = c_k - c_{k-1}$. Hence, $c_k \in \mathbb{N}$ for all $k \in \mathbb{N}$ if and only if

$$\lambda_1 + \cdots + \lambda_k = c_1 + (c_2 - c_1) + \cdots + (c_k - c_{k-1}) = c_k \geq 0.$$

For dominant weights $\lambda, \mu \in \Lambda^+(G)$, the dominance order is in agreement with the following combinatorial definition of dominance order for integer partitions: Two integer partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ satisfy $\lambda \trianglerighteq \mu$ if and only if $|\lambda| = |\mu|$ and for all $k \leq n$, we have $\lambda_1 + \cdots + \lambda_k \geq \mu_1 + \cdots + \mu_k$. ●

Finally, we sum up the characterization of the irreducible representations of G in the following theorem:

A.2.6 Theorem. Let $\varrho \in \mathrm{Irr}(G)$ and $V := \mathbb{V}(\varrho)$. There exists a unique dominant weight $\lambda \in \Lambda^+(G)$ such that:

(1) As a T-module, $V = \bigoplus_{\mu \trianglelefteq \lambda} V_\mu$.

(2) $\dim(V_\lambda) = 1$. Elements of V_λ are called **highest weight vectors**.

(3) U acts trivially on V_μ if and only if $\mu = \lambda$.

(4) For $\alpha \in \Phi$ and $\mu \in \Lambda(G)$, we have $U_\alpha.V_\mu \subseteq \bigoplus_{d \in \mathbb{N}} V_{\mu + d\alpha}$.

This defines a bijection $\mathrm{Irr}(G) \cong \Lambda^+(G)$. We henceforth identify the two.

Proof. The main statement as well as (1) to (3) can be found in [Hum98, Theorem 31.3] together with the statement [Hum98, 31.4] which proves that each dominant weight has a corresponding irreducible representation. Part (4) is [Hum98, 27.2]. ∎

For $\varrho \in \mathrm{Rep}(G)$, the module $V := \mathbb{V}(\varrho)$ decomposes as

$$V = \bigoplus_{\lambda \in \Lambda^+(G)} V_{(\lambda)}$$

where $V_{(\lambda)} \cong \mathbb{V}(\lambda)^{\oplus n_\lambda}$ for certain $n_\lambda \in \mathbb{N}$. We call $V_{(\lambda)}$ the **isotypical component** of weight λ of V. When $f \in V_{(\lambda)}$, we call $\mathrm{wt}(f) := \lambda$ the **weight** of f.

A.2.3 The Coordinate Ring of an Algebraic Group

We recite a famous, general result about the weights that appear in the coordinate ring of a reductive, affine, algebraic group:

A.2.7 Theorem (Algebraic Peter-Weyl Theorem). Let G be a reductive, affine, algebraic group. The group $G \times G$ acts on G from the left via $(g,h).x = gxh^{-1}$. Consider

$$\begin{aligned} \psi\colon \bigoplus_{\lambda \in \mathrm{Irr}(G)} \mathbb{V}(\lambda)^* \otimes \mathbb{V}(\lambda) &\longrightarrow \mathbb{C}[G] \\ f \otimes v &\longmapsto f \circ \omega_v \end{aligned}$$

where $\omega_v\colon G \to \mathbb{V}(\lambda)$ is the morphism $g \mapsto g.v$.

Then, ψ is an isomorphism of $G \times G$ – modules, where the action of the left and right factor of $G \times G$ is on the left and right tensor factor, respectively.

Proof. See [TY05, 27.3.9]. We give a quick remark about the action. Recall from (1) that $g \in G$ acts on $f \in \mathbb{V}(\lambda)^*$ by $g.f = f \circ g^{-1}$ and recall from Remark A.1.5 that the induced action on the coordinate ring of G is such that for $\varphi \in \mathbb{C}[G]$ and $g, h, x \in G$

we have $((g,h).\varphi)(x) = \varphi(g^{-1}xh)$. Let $\lambda \in \mathrm{Irr}(G)$, $f \in \mathbb{V}(\lambda)^*$, $v \in \mathbb{V}(\lambda)$ and $g, h \in G$. Then, the following holds for all $x \in G$:

$$\begin{aligned}((g,h).\psi(f \otimes v))(x) &= ((g,h).(f \circ \omega_v))(x) = f(\omega_v(g^{-1}xh)) = f(g^{-1}xh.v) \\ &= (f \circ g^{-1})(x.(h.v)) = ((f \circ g^{-1}) \circ \alpha_{h.v})(x) = \psi((g.f) \otimes (h.v))(x).\end{aligned}$$

Hence, $(g,h).\psi(f \otimes v) = \psi((g.f) \otimes (h.v))$ as claimed. ■

A.2.8. We summarize Examples A.2.1 and A.2.3 to A.2.5 as follows: The dominant weights for the reductive group GL_n are the **generalized partitions**

$$\Lambda_n^+ := \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\}.$$

We denote by $\mathbb{V}(\lambda)$ the irreducible GL_n-module corresponding to the highest weight $\lambda \in \Lambda_n^+$. Every GL_n-module $\mathbb{V}$ decomposes as $\mathbb{V} = \bigoplus_{\lambda \in \Lambda_n^+} \mathbb{V}(\lambda)^{n_\lambda}$ for certain $n_\lambda \in \mathbb{N}$. ●

A.2.9 Proposition. The coordinate ring of $\mathrm{GL}_n \subseteq \mathbb{C}^{n \times n}$ has a natural $\mathbb{Z}$-grading because it is the localization of a polynomial ring at the homogeneous polynomial $\det_n$. For $d \in \mathbb{Z}$, the isomorphism from Theorem A.2.7 restricts to

$$\mathbb{C}[\mathrm{GL}_n]_d \cong \bigoplus_{\substack{\lambda \in \Lambda_n^+ \\ |\lambda| = d}} \mathbb{V}(\lambda)^* \otimes \mathbb{V}(\lambda).$$

Proof. Let $\lambda \in \Lambda_n^+$, $d := |\lambda|$ and $f \otimes v \in \mathbb{V}(\lambda)^* \otimes \mathbb{V}(\lambda)$. By Theorem A.2.7 we only have to show that $f \circ \omega_v$ is homogeneous of degree d. Given $t \in \mathbb{C}^\times$, Theorem A.2.6 implies that $\delta_t := \mathrm{diag}(t, \ldots, t)$ acts on a vector $v \in \mathbb{V}(\lambda)$ as $\delta_t.v = t^d \cdot v$ because for all $\mu \trianglelefteq \lambda$, we have $|\mu| = |\lambda| = d$ (Example A.2.5). Hence,

$$\begin{aligned}f(\omega_v(t \cdot g)) &= f(\omega_v(g\delta_t)) = f(g.\delta_t.v) = f(g.(t^d \cdot v)) = f(t^d \cdot (g.v)) \\ &= t^d \cdot f(g.v) = t^d \cdot f(\omega_v(g)).\end{aligned}$$

Note that the fourth and fifth equality are due to the fact that $v \mapsto g.v$ and f are both linear maps. ■

The module $\mathbb{V}(\lambda)^*$ is also an irreducible module. More precisely, for any dominant weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_n^+$ we define $\lambda^* := (-\lambda_n, \ldots, -\lambda_1) \in \Lambda_n^+$. Then,

A.2.10 Proposition. Let $\lambda \in \Lambda_n^+$. Then, $\mathbb{V}(\lambda)^* = \mathbb{V}(\lambda^*)$ is an irreducible module.

Proof. This follows from Theorem A.2.6 and (1). ■

A.2.11. Proposition A.2.10 is a special case of a general principle. We denote by $\mathfrak{B}$ the set of all Borel subgroups of G. By [Hum98, Proposition A in 24.1], the Normalizer $\mathfrak{N} := \mathrm{N}_G(T)$ of T permutes the elements of $\mathfrak{B}$ transitively. For $G = \mathrm{GL}_n$ and T the subgroup of diagonal invertible matrices, note that $\mathfrak{N}$ is the semidirect product $\mathfrak{S}_n \ltimes T$, where the permutation group $\mathfrak{S}_n$ is embedded in GL_n as the subgroup of permutation matrices.

The induced action of the **Weyl group** $\mathfrak{S} := \mathfrak{N}/T$ is free, so $\mathfrak{S}$ parametrizes exactly the possible choices of a base for the root system with respect to the torus T. The action of $\mathfrak{S}$ on $\mathfrak{B}$ is given by $B \mapsto B^q := q^{-1}Bq$ for $q \in \mathfrak{N}$.

Let $\tau \in \mathfrak{N}$ be such that $B \cap B^\tau = T$. Note that in the case of $G = \mathrm{GL}_n$, this is simply the permutation that maps $(1, \ldots, n)$ to the reverse vector $(n, \ldots, 1)$, because B is the set of upper triangular matrices and $B \cap B^\tau = T$ if and only if B^τ is the set of *lower* triangular matrices.

The image of τ in $\mathfrak{S}$ is called the *longest Weyl element*. In the literature, it is often denoted by w_0. With the terminology from [Spr08], w_0^{-1} has the same *length* as w_0 and by [Spr08, Corollary 8.3.11], an element with this property is unique, hence $w_0 = w_0^{-1}$. It follows that $\tau = \tau^{-1}$.

The Weyl group acts on the weights $\mathbb{X}(T)$ as follows: For any $\chi \in \mathbb{X}(T)$ and $q \in \mathfrak{N}$, we define the character $q.\lambda$ via the rule $(q.\lambda)(t) := \lambda(qtq^{-1})$. Then, the longest Weyl element maps highest weights to highest weights [Hum98, 31.6]. More precisely, we have $\mathbb{V}(\lambda)^* = \mathbb{V}(-\tau.\lambda)$. In the case of $G = \mathrm{GL}_n$, we have $-\tau.\lambda = \lambda^*$. ●

In particular:

A.2.12 Proposition. Let V be a G-module. Then, $V^G \cong (V^*)^G$.

Proof. By Theorem A.2.6, $V = \bigoplus_{\lambda \in \Lambda^+(G)} V_{(\lambda)}$, so $V^G = V_{(0)}$. By the above Paragraph A.2.11, $V^* = \bigoplus_{\lambda \in \Lambda^+(G)} V_{(-\tau.\lambda)}$ where τ denotes the longest Weyl element. Since $-\tau.\lambda = 0$ is equivalent to $\lambda = 0$, we have $(V^*)^G \cong V^G$. ■

We end by proving a Lemma that we will require in the next section and which is of independent interest:

A.2.13 Lemma. Let V be a G-module (or more generally, a factorial G-variety) and $f \in \mathbb{C}[V]$. Assume that $f = f_1^{k_1} \cdots f_r^{k_r}$ is the decomposition of f into irreducible factors. Then, f is a highest weight vector of weight λ if and only if the f_i are highest weight vectors of weight λ_i with $\lambda = k_1\lambda_1 + \cdots + k_r\lambda_r$.

Proof. If the f_i are weight vectors, then f is still U-invariant and it is easy to see that the weight of f must be equal to $k_1\lambda_1 + \cdots + k_r\lambda_r$.

For the converse, assume that f is a highest weight vector of weight λ and denote by $Z := \mathrm{Z}(f) \subseteq V$ the vanishing set of f. Note that Z is B-invariant, so we have an action $\alpha\colon B \times Z \to Z$ induced by the action of B on V. Let $Z_i := \mathrm{Z}(f_i)$, then $Z = Z_1 \cup \ldots \cup Z_r$ is the decomposition of Z into its irreducible components. Let $\alpha_i\colon B \times Z_i \to Z$ be the restriction of α to Z_i. Since B is connected, $B \times Z_i$ is irreducible and Z_i is contained in the image of α_i – therefore, $B.Z_i = \alpha_i(B \times Z_i) = Z_i$. Hence, the Z_i are invariant under B. It follows that $B.f_i \subseteq \mathbb{C} \cdot f_i$ so B acts by a character on f_i. This means that f_i is a highest weight vector of some weight λ_i. Consequently, we must have $\lambda = k_1\lambda_1 + \cdots + k_r\lambda_r$. ■

A.3 Polynomial Representations

Let $W \cong \mathbb{C}^n$ be a complex vector space with a chosen basis. The action of GL_n on a module $\mathbb{V} \cong \mathbb{C}^N$ is a morphism $\varrho\colon \mathrm{GL}_n \to \mathrm{GL}_N$ of algebraic groups. Hence, it can be described by N^2 regular functions on GL_n. In many cases we are interested in, these functions are polynomials in the entries of an $n \times n$ matrix, i.e., ϱ extends to a morphism $\mathrm{End}_n \to \mathrm{End}_N$ of affine spaces. In other words, the action of GL_n extends to an action $\mathbb{V} \times \mathrm{End}_n \to \mathbb{V}$. We call such a representation a **polynomial** one.

A.3.1 Definition. There is a partial ordering $\sqsubseteq$ on Λ_n^+, called the **inclusion**, defined as follows: We have $\mu \sqsubseteq \lambda$ if and only if $\mu_i \leq \lambda_i$ for all $1 \leq i \leq n$. ●

Note that $\mu \sqsubseteq \lambda$ implies $\mu \trianglelefteq \lambda$, but not the converse. We have

$$\underline{\Lambda}_n^+ := \{\lambda \in \Lambda_n^+ \mid \lambda \sqsupseteq 0\} = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \ldots \geq \lambda_n \geq 0\} \subseteq \mathbb{N}^n,$$

the set of **partitions** of the number n. We will show:

A.3.2 Proposition. Let $\lambda \in \Lambda_n^+$, then $\mathbb{V}(\lambda)$ is polynomial if and only if $\lambda \in \underline{\Lambda}_n^+$.

When $\mathbb{V}$ is any $\mathrm{GL}(W)$-module, we can decompose it as $\mathbb{V} \cong \bigoplus_{\lambda \in \Lambda} \mathbb{V}(\lambda)^{\oplus n_\lambda}$ for certain $n_\lambda \in \mathbb{N}$. We then write

$$\mathbb{V}_{\sqsupseteq 0} := \bigoplus_{\substack{\lambda \in \Lambda_n^+ \\ \lambda \sqsupseteq 0}} \mathbb{V}(\lambda)^{\oplus n_\lambda} \subseteq \mathbb{V}.$$

We call $\mathbb{V}_{\sqsupseteq 0}$ the **polynomial part** of $\mathbb{V}$. It then follows:

A.3.3 Corollary. A GL_n-module $\mathbb{V}$ is polynomial if and only if $\mathbb{V} = \mathbb{V}_{\sqsupseteq 0}$. ■

We require an auxiliary lemma for the proof of Proposition A.3.2.

A.3.4 Lemma. The inclusion $\mathrm{GL}_n \subseteq \mathrm{End}_n$ is an open, GL_n-equivariant immersion of varieties under the operation of GL_n acting by multiplication from the left on both affine varieties. It induces an inclusion of their respective coordinate rings which satisfies $\mathbb{C}[\mathrm{End}_n] = \mathbb{C}[\mathrm{GL}_n]_{\sqsupseteq 0} = \bigoplus_{\lambda \in \underline{\Lambda}_n^+} \mathbb{V}(\lambda) \otimes \mathbb{V}(\lambda)^*$.

Proof. By [Lan12, Formula (6.5.1)], we get the last equality in

$$\mathbb{C}[\mathrm{End}(W)]_d \cong \mathbb{C}[W \otimes W^*]_d \cong \mathrm{Sym}^d(W \otimes W^*) \cong \bigoplus_{\substack{0 \sqsubseteq \lambda \in \Lambda_n^+ \\ \lambda_1 + \cdots + \lambda_n = d}} \mathbb{V}(\lambda) \otimes \mathbb{V}(\lambda)^*.$$

By summing over d and applying Theorem A.2.7, we obtain

$$\mathbb{C}[\mathrm{End}(W)] \cong \bigoplus_{\lambda \in \underline{\Lambda}_n^+} \mathbb{V}(\lambda) \otimes \mathbb{V}(\lambda)^* = \mathbb{C}[\mathrm{GL}(W)]_{\sqsupseteq 0}$$

by the definition of the polynomial part. ∎

Proof of Proposition A.3.2. By Theorem A.2.7, the module $\mathbb{C}[\mathrm{GL}_n]$ contains a *unique* highest weight vector of weight μ for every $\mu \in \Lambda_n^+$, up to a scalar. Let f be the highest weight vector of weight $\lambda = (\lambda_1, \dots, \lambda_n)$. By Lemma A.3.4, we have $\lambda \in \underline{\Lambda}_n^+$ if and only if f extends to a regular function on End_n, i.e., it is a polynomial. Let $f = q \cdot \det_n^k$ with $k \in \mathbb{Z}$ and $q \in \mathbb{C}[\mathrm{End}_n]$ not divisible by $\det_n$. We have to show that $k \geq 0$ is equivalent to $\lambda_n \geq 0$. In fact, we show that $k = \lambda_n$.

Denote by $\delta := (1, \dots, 1) \in \Lambda_n^+$, then $\mathrm{wt}(\det_n) = \delta$. We claim that $\mu := \mathrm{wt}(q)$ satisfies $\mu_n = 0$. Indeed, otherwise we'dhave $\mu - \delta \in \underline{\Lambda}_n^+$ and Lemma A.2.13 would imply that q is divisible by the (unique) weight vector of weight δ, which is $\det_n$. Consequently, $\lambda_n = \mu_n + k = k$. ∎

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List of Symbols

$\sqsubseteq$	Inclusion order on generalized partitions (page 137).
$\equiv$	Equivalence of complexity measures (page 36).
$\nabla(P)$	Row vector of partial derivatives of P (page 59).
$[m]$	The set of natural numbers from 1 to m (page 16).
$X /\!\!/ G$	Quotient of a G-variety X by G (page 125).
$P \sim Q$	The form P is in the GL-orbit of the form Q. (page 108).
α_X	action morphism of a variety with semigroup action (page 123).
$A^\sharp$	Adjugate of A, transpose of the cofactor matrix (page 80).
$\hat{\mathcal{A}}_P$	Annihilator scheme (page 82).
$\hat{\mathcal{A}}_P^{\mathrm{ss}}$	Semistable part of $\hat{\mathcal{A}}_P$ (page 82).
A^{t}	Transpose of the matrix A (page 41).
$\mathcal{A}_P$	Annihilator of a form P (page 61).
$\mathrm{bdc}(P)$	Binary determinantal complexity of a polynomial P (page 21).
bn_d	The generic binomial (page 105).
$\mathrm{cc}(P)$	Circuit complexity of a polynomial P with coefficients in a fixed ring (usually a field). (page 10).
$\mathrm{cc}^R(P)$	Circuit complexity of a polynomial P with coefficients in R (page 10).
$\mathscr{C}_d$	d-th Chow variety, orbit closure of $\mathrm{mn}_d = x_1 x \dots x_d$ (page 49).
$\mathrm{cone}_{\mathbb{Q}}(S)$	Rational cone generated by a subset $S \subseteq \mathbb{Z}^n$ (page 47).
$\mathbb{C}[\![t]\!]$	The ring of formal power series with complex coefficients (page 77).
$\mathbb{C}[X]$	Coordinate ring of an affine variety X (page 123).
$\mathscr{D}_d$	The orbit closure of $\det_d$ (page 45).
D_ϱ	Differential of a morphism ϱ between varieties (page 79).

$\partial_i P$	The i-th partial derivative of a polynomial P (page 58).
$\partial_w P$	Partial derivative of a form P in direction w (page 58).
$\mathscr{P}_{d,m}$	The orbit closure of $\mathrm{pp}_{d,m}$ (page 45).
$\mathrm{dc}(P)$	Determinantal complexity of a polynomial P (page 18).
$\underline{\mathrm{dc}}(P)$	Determinantal border complexity of a polynomial P (page 37).
$\det_d$	The determinant polynomial of a generic $d \times d$ matrix (page 16).
$\mathrm{doc}(P)$	Determinantal orbit complexity of a polynomial P (page 36).
$\underline{\mathrm{doc}}(P)$	Determinantal orbit border complexity of a polynomial P (page 37).
$\mathrm{dom}(\varpi)$	Domain of the rational map ϖ (page 81).
$\mathrm{End}(W)$	The linear endomorphisms of a vector space W (page 35).
$\mathrm{End}(W, L)$	Endomorphisms on W that map into the subspace $L \subseteq W$ (page 89).
G°	Identity component of an algebraic group G (page 125).
$\Gamma(\phi)$	Graph of a rational map ϕ (page 81).
G_P	Stabilizer of a homogeneous polynomial P (page 41).
$\mathbb{G}_\mathrm{a}$	The additive group (page 128).
$\mathrm{GL}(W)$	The general linear group of invertible endomorphisms of W (page 124).
GL_n	The group of invertible, complex $n \times n$ matrices (page 124).
$\mathfrak{gl}_n$	Lie algebra of GL_n (page 79).
$\mathfrak{gl}(V)$	Lie algebra of $\mathrm{GL}(V)$ (page 79).
$\mathbb{G}_\mathrm{m}$	The multiplicaive group (page 128).
HC_m	Hamilton Cycle polynomial (page 31).
$\mathbb{I}_n$	The $n \times n$ identity matrix (page 47).
id_W	The identity map on W (page 59).
$\mathrm{im}(a)$	Image of an endomorphism a (page 63).
$\mathrm{Irr}(G)$	All equivalence classes of irreducible G-representations (page 130).
$\lvert\lambda\rvert$	Sum of all entries of an integer vector $\lambda \in \mathbb{Z}^n$ (page 46).
$\ell(\lambda)$	Length of a partition λ (page 46).
L	Usually a linear subspace of W, often a linear subspace of some hypersurface $\mathrm{Z}(P) \subseteq W$ (page 61).
λ^*	Negative reverse of a parttion λ (page 135).
$\Lambda^+(G)$	Dominant weights of an affine algebraic group G (page 132).

$\Lambda^+(Z)$ Dominant weights in the coordinate ring of a GL_n-variety Z (page 46).

$\Lambda(G)$ Weight lattice of an affine algebraic group G (page 131).

$\mathfrak{Lie}(G)$ Lie algebra of an algebraic group G (page 79).

Λ_n^+ Generalized partitions, the dominant weights of GL_n (page 135).

$\underline{\Lambda}_n^+$ Dominant weights of GL_n that yield polynomial representations, partitions of n (page 137).

$\Lambda(Z)$ Group generated by $\Lambda^+(Z)$ (page 47).

mn_d The universal monomial $\mathrm{mn}_d = x_1 \cdots x_d$ (page 49).

$\mathcal{N}_P$ Nullcone of the action of the stabilizer group on the space of Endomorphisms (page 62).

$\mathrm{N}(Z)$ Normalization of a variety Z (page 48).

Ω_P, $\Omega(P)$ Orbit of a homogeneous polynomial (page 39).

$\mathrm{oc}_P(\lambda)$ Orbit coefficient of P at λ (page 42).

$\overline{\Omega}_P$, $\overline{\Omega}(P)$ Orbit closure of a homogeneous polynomial (page 39).

$\underline{\mathrm{oc}}_P(\lambda)$ Orbit closure coefficient of P at λ (page 40).

$\partial\Omega_P$ Boundary of the orbit of a homogeneous form (page 60).

$\mathcal{P}$ Usually a set of polynomials (page 35).

Φ Usually a root system (page 131).

per_m The permanent polynomial of a generic $m \times m$ matrix (page 16).

$P \leq Q$ P is a projection of Q over a fixed ring (page 14).

$P \simeq Q$ The two polynomials P and Q are equivalent over a fixed ring (page 35).

$P \leq_R Q$ P is a projection of Q over the ring R (page 14).

$P \simeq_R Q$ The two polynomials P and Q are equivalent over the ring R (page 35).

poly The set of polynomially bounded functions $\mathbb{N} \to \mathbb{N}$. (page 12).

$\mathrm{poly}(n)$ Expressions that are polynomially bounded in $n \in \mathbb{N}$. (page 12).

$\mathrm{pp}_{d,m}$ Padded permanent polynomial (page 39).

P^σ The polynomial that arises from P by the substitution σ (page 14).

$\mathcal{Q}$ Usually a set of polynomials (page 35).

$\mathrm{Rep}(G)$ All equivalence classes of G-representations (page 130).

$\mathrm{R}(G)$ Radical of an algebraic group G (page 128).

$\mathrm{rk}(a)$ Rank of an endomorphism a (page 59).

$\mathrm{R_u}(G)$	Unipotent radical of an algebraic group G (page 128).
$R[\mathbf{x}]$	The polynomial ring in a countably infinite set $\mathbf{x}$ of variables (page 10).
$\langle S \rangle$	Free abelian group generated by S (page 46).
$\mathfrak{S}_m$	The symmetric group on m symbols (page 16).
$\mathrm{Sat}(S)$	Saturation of a semigroup S (page 46).
SL_n	Group of complex $n \times n$ matrices with determinant 1, the special linear group (page 129).
$\mathfrak{sl}_n$	Lie algebra of SL_n (page 81).
$\mathrm{supp}(P)$	Support of a polynomial P (page 13).
$\mathrm{Sym}^k W$	The k-th symmetric power of the vector space W (page 49).
$\mathrm{tr}(A)$	The trace of a square matrix A (page 80).
V	A complex vector space, usually $V = \mathbb{C}[W]_d$ (page 57).
$\mathbb{V}_{\sqsupseteq 0}$	Polynomial part of a GL_n-module $\mathbb{V}$ (page 137).
$\mathbb{V}(\varrho)$	Vector space of a representation ϱ (page 129).
$\mathbb{V}_G(\varrho)$	Vector space of a G-representation ϱ (page 129).
ω_P	Orbit map of a point/polynomial P (page 61).
ϖ_P	Rational projective orbit map of a point/polynomial P (page 61).
W	A complex vector space of dimension n (page 57).
$\mathbf{x}$	A countably infinite set of formal variables (page 10).
$\mathbb{X}(G)$	Character group of G (page 130).
$\check{\mathbb{X}}(G)$	One-paramter-subgrups of G (page 131).
$\mathrm{Z}(P)$	Affine vanishing set of a homogeneous form P (page 63).

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